MAST31005 Algebra II exercise 1 (24.01.2024)

1. Let $K \hookrightarrow L$ be a field extension. Verify the details omitted in the lectures that L is a vector space over K.

2. Let $\mathbb{Q}(x_1, \dots, x_n)$ be the fraction field in the indeterminates $x_1, \dots, x_n$ and let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Show that the field extension $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ consists of

$$\mathbb{Q}(\alpha_1, \dots, \alpha_n) = \left\{ \frac{p(\alpha_1, \dots, \alpha_n)}{q(\alpha_1, \dots, \alpha_n)} : \frac{p}{q} \in \mathbb{Q}(x_1, \dots, x_n) \right\}.$$

3. Let K be a subfield of $\mathbb{C}$. Let $\alpha \in \mathbb{C}$ be algebraic over K and let $p \in K[t]$ be the minimal polynomial of α. If $\deg(p) = 1$, show that $\alpha \in K$.

4. Find the minimal polynomial of $1 + i \in \mathbb{C}$ over $\mathbb{Q}$.

5. Show that $[\mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15}) : \mathbb{Q}] = 4$.

6. Show that $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are linearly independent over $\mathbb{Q}$. You may assume that we know $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Hint: consider a $\mathbb{Q}$-linear relation $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = 0$. Argue that a nontrivial relation would have to have either $c \neq 0$ or $d \neq 0$. Use this assumption to write $\sqrt{3}$ in the form $\sqrt{3} = f + g\sqrt{2}$ with $f, g \in \mathbb{Q}$ and derive a contradiction.

7. Let K be a subfield of $\mathbb{C}$ and $p \in K[t]$ an irreducible polynomial. Let $\alpha \in \mathbb{C}$ be some root of p. Does p factorize into linear terms over $K(\alpha)$? Hint: consider $K = \mathbb{Q}$, $\alpha = \sqrt[3]{2} \in \mathbb{R}$.