MAST31005 Algebra II exercise 12 (24.04.2024)

1. Let K be an infinite field.

(a) Show that the coordinate ring $K[V]$ of the variety $V = K$ and the polynomial ring $K[x]$ are isomorphic rings.

(b) Show that the coordinate ring $K[V]$ of the variety $V = K^n$ and the polynomial ring $K[x_1, \dots, x_n]$ are isomorphic rings.

(c) What goes wrong when K is not infinite?

2. (a) Show that $\phi(x, y, z) = (2x^2+y^2, z^2-y^3+3xz)$ and $\psi(x, y, z) = (2y+xz, 3y^2)$ represent the same polynomial mapping $V(y - x^2, z - x^3) \to \mathbb{R}^2$.

(b) Find another representation of this same polynomial mapping with all components in the same univariate polynomial ring.

3. Let $V = V(y - x) \subset \mathbb{R}^2$. Let $\phi\colon V \to \mathbb{R}^3$ be a polynomial mapping represented by $\phi(x, y) = (x^2 - y, y^2, x - 3y^2)$. The image $W = \phi(V) \subset \mathbb{R}^3$ is a variety. Find a system of equations defining the variety W.

4. Let $W = V(y^2 - x^3 + x) \subset \mathbb{R}^2$. Show that there are no non-constant polynomial mappings $\mathbb{R} \to W$ using the following argument (or any other argument!).

(a) Let $\phi\colon \mathbb{R} \to W$ be a polynomial mapping represented by $\phi(t) = (a(t), b(t))$. Show that $b(t)^2 = a(t)(a(t)^2 - 1)$.

(b) Show that the factors $a(t)$ and $a(t)^2 - 1$ must be relatively prime in $\mathbb{R}[t]$.

(c) Using factorizations of a and b into irreducibles, show that $b^2 = ac^2$ for some $c \in \mathbb{R}[t]$ with a and c relatively prime.

(d) Deduce that $c^2 = a^2 - 1$ implies that a, b, and c must be constant polynomials.

5. Let $V = V(y-x^n, z-x^m) \subset K^3$, where $n, m \geq 1$ are arbitrary positive integers. Show that V is isomorphic to K by constucting explicit inverse polynomial mappings $\alpha\colon K \to V$ and $\beta\colon V \to K$.

6. Let $V \subset K^n$ be a variety defined as a graph of a polynomial mapping $p\colon K^{n-1} \to K$, i.e. V defined by the equation $x_n - p(x_1, \dots, x_{n-1}) = 0$. Show that V is isomorphic to K^{n-1}.

7. Let $I = \langle x_3 - x_1^2, x_4 - x_1x_2, x_2x_4 - x_1x_5, x_4^2 - x_3x_5\rangle \subset \mathbb{C}[x_1, \dots, x_5]$ and $V = V(I) \subset \mathbb{C}^5$.

(a) Find a collection of monomials such that their $\mathbb{C}$-linear span is isomorphic to $K[V]$ as a $\mathbb{C}$-vector space.

(b) For which $i \in \{1, \dots, 5\}$ does there exist $m_i \geq 0$ such that $x_i^{m_i} \in I$?

(c) Is V finite?