MAST31005 Algebra II exercise 13 (02.05.2024)

1. Let $I = \langle x^2 - 2xy - 4x + 2yz + 6y - 3z, y^2 - 2yz - 2y + 3z, z^2 - z \rangle$. Show that I is zero dimensional and find the points of $V(I)$. What does the tree-structure of expanding partial solutions into full solutions as in Example 10.19 look like?

2. Let $V \subset \mathbb{C}^n$ be a nonempty variety and $\phi \in \mathbb{C}[V]$. Show that $V_V(\phi) = \emptyset$ if and only if $1/\phi \in \mathbb{C}[V]$, i.e., there exists $\psi \in \mathbb{C}[V]$ such that $\phi\psi = 1$.

3. Let $\alpha \colon V \to W$ and $\beta \colon W \to V$ be inverse polynomial mappings. Let $U = V_V(I)$ for some ideal $I \subset K[V]$. Find an ideal $J \subset K[W]$ such that $\alpha(U) = V_W(J)$.

4. Let $V = V(xy - 1) \subset \mathbb{C}^2$ and $W = V(xyz - 1) \subset \mathbb{C}^3$. Find non-constant rational mappings $\phi \colon V \dashrightarrow W$ and $\psi \colon W \dashrightarrow \mathbb{C}$ such that the composition $\psi \circ \phi$ is not defined.

5. Show that $V(x^2 + y^2 + z^2 - 1) \subset \mathbb{R}^3$ and $\mathbb{R}^2$ are birationally equivalent via the stereographic projection

$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

and its inverse

$$(u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

6. Show that $V(y^2 - x^3) \subset K^2$ and K are birationally equivalent.