MAST31005 Algebra II exercise 2 (31.01.2024)

1. Let $K \subset \mathbb{C}$ be a subfield. Let $p, q \in K[t]$ and $\alpha \in \mathbb{C}$ be such that $p(\alpha) = 0 = q(\alpha)$. Show that there exists a polynomial $f \in K[t]$ such that $\deg f \geq 1$ and $f \mid p$ and $f \mid q$.

2. Let $K \hookrightarrow L$ be a finite extension, and p an irreducible polynomial over K. Suppose $\deg p$ does not divide $[L : K]$. Show that p has no roots in L.

3. Let $K \subset \mathbb{C}$ be a subfield and $0 \neq \alpha \in \mathbb{C}$ algebraic over K.

(a) Show that the minimal polynomial $m \in K[t]$ of α satisfies $m(0) \neq 0$.

(b) Let $p \in K[t]$ be a monic polynomial with $\deg p \geq 1$. Show that the companion matrix $C(p)$ is invertible if and only if $p(0) \neq 0$.

(c) Conclude that there exists an invertible matrix $A \in K^{n\times n}$ with eigenvalue α.

4. Let $K \hookrightarrow L$ be a field extension with $[L : K] = 2$. Show that $L = K(\alpha)$ for some element $\alpha \in L$ with $\alpha^2 \in K$.

5. Show that the field of algebraic numbers $\overline{\mathbb{Q}} \subset \mathbb{C}$ has no nontrivial finite extensions.

6. Let $\alpha = \sqrt[4]{2}$ and $\beta = \sqrt[20]{2}$ with minimal polynomials $p_\alpha = t^4 - 2$ and $p_\beta = t^{20} - 2$ over $\mathbb{Q}$. Let $A = C(p_\alpha) \in \mathbb{Q}^{4\times 4}$ and $B = C(p_\beta) \in \mathbb{Q}^{20\times 20}$ be the respective companion matrices. Let $f = \det(tI - A \otimes B) \in \mathbb{Q}[t]$ be the characteristic polynomial of the Kronecker product $A \otimes B \in \mathbb{Q}^{80\times 80}$. Show that f is not the minimal polynomial of the product $\alpha\beta$.

7. The map

$$\Phi\colon \mathbb{Q}[t] \times \mathbb{Q}[t] \to \mathbb{Q}[t], \quad \Phi(p, q) = \det(tI - C(p) \otimes I - I \otimes C(q))$$

has the "sum of roots" property: if $\alpha \in \mathbb{C}$ is a root of $p \in \mathbb{Q}[t]$ and $\beta \in \mathbb{C}$ is a root of $q \in \mathbb{Q}[t]$, then $\alpha + \beta \in \mathbb{C}$ is a root of $\Phi(p, q) \in \mathbb{Q}[t]$.

Let $S \subset \mathbb{Q}[t]$ be the subset of all irreducible polynomials. Show that there cannot exist a map $\Psi\colon S \times S \to S$ with the "sum of roots" property.