MAST31005 Algebra II exercise 5 (21.02.2024)

1. Prove the following ascending/descending chain conditions or give a counterexample:

(a) DCC for ideals: If $K[x_1, \dots, x_n] \supset I_1 \supset I_2 \supset I_3 \supset \cdots$ ideals, then $I_N = I_{N+1} = \dots$ for some $N \geq 1$.

(b) ACC for varieties: If $V_1 \subset V_2 \subset V_3 \subset \cdots \subset K^n$ varieties, then $V_N = V_{N+1} = \dots$ for some $N \geq 1$.

(c) DCC for varieties: If $K^n \supset V_1 \supset V_2 \supset V_3 \supset \dots$ varieties, then $V_N = V_{N+1} = \dots$ for some $N \geq 1$.

Hint: For varieties, consider the ideal $I(V)$. For counterexamples it suffices to consider the univariate case $K[x]$.

2. Let $V \subset K^n$ be a variety. Let $W = V(I(V))$ be the variety of the ideal of the variety V. Show that $W = V$.

3. Let $I = \langle p \rangle \subset K[x_1, \dots, x_n]$. Show that $\{p\}$ is a Gröbner basis of I with respect to any monomial order.

4. Let $I \subset K[x_1, \dots, x_n]$ be an ideal and $G \subset I$ a basis such that for all $f \in I$ the remainder of division by G is $\overline{f}^G = 0$. Show that G is a Gröbner basis of I.

5. Let $A = (a_{ij}) \in K^{m \times n}$ be a matrix in row echelon form. Let $g_i = \sum_{j=1}^{n} a_{ij} x_j \in K[x_1, \dots, x_n]$ be the polynomials given by the rows of A. Show that there is a monomial order on $K[x_1, \dots, x_n]$ such that $\{g_1, \dots, g_m\}$ is a Gröbner basis of the ideal $\langle g_1, \dots, g_m \rangle$.

6. Consider lex order on $\mathbb{R}[x, y, z]$ and let $g_1 = x + z$ and $g_2 = y + z$.

(a) Using Buchberger's criterion, verify that g_1, g_2 is a Gröbner basis of $\langle g_1, g_2 \rangle$.

(b) Show that the division algorithm gives different polynomials q_1, q_2 when dividing xy by (g_1, g_2) or by (g_2, g_1).

7. Compute a Gröbner basis for $I = \langle y^2 + x, y^4 + 2xy^2 + x^2 + 3 \rangle$ in the deglex order. What can you deduce about the variety $V(I)$?

8. Let $V = \{(b_1, a_1), \dots, (b_n, a_n)\} \subset K^2$ with $a_i \neq a_j$ for $i \neq j$. Consider the polynomial with roots a_i

$$f = \prod_{i=1}^{n} (x - a_i) \in K[x]$$

and the Lagrange interpolation polynomial

$$h = \sum_{i=1}^{n} b_i \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \in K[x].$$

(a) Show that h is the unique polynomial in $K[x]$ with $\deg h \leq n - 1$ and $h(a_i) = b_i$.

(b) Show that $\{f, y - h\} \subset K[y, x]$ is a Gröbner basis for $I(V) \subset K[y, x]$ in the lex order.