MAST31005 Algebra II exercise 6 (28.02.2024)

For computations, use of a computer algebra system is recommended. When using a CAS, explain which system you used, how you implemented the problem (math-to-computer translation), and how you interpreted the output (computer-to-math translation).

1. Is $f = xy^3 - z^2 + y^5 - z^3$ contained in the ideal $I = \langle -x^3 + y, x^2y - z\rangle$?

2. Find all the points of the variety

$$V(x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z) \subset \mathbb{C}^3.$$

3. Find all the points of the variety

$$V(x^2y - z^3, 2xy - 4z - 1, -y^2 + z, x^3 - 4yz) \subset \mathbb{C}^3.$$

4. Suppose $a, b, c \in \mathbb{C}$ are such that

$$a + b + c = 3 \tag{1}$$
$$a^2 + b^2 + c^2 = 5 \tag{2}$$
$$a^3 + b^3 + c^3 = 7 \tag{3}$$

Show that

(i) $a^4 + b^4 + c^4 = 9$,

(ii) $a^5 + b^5 + c^5 \neq 11$.

(iii) What is the value of $a^6 + b^6 + c^6 \in \mathbb{C}$?

5. In exercise 4, instead of $\mathbb{C}$ consider a subfield $K \subset \mathbb{C}$. For which subfields $K \subset \mathbb{C}$ does one or both of the implications

$$(1), (2), (3) \implies \text{(i)}$$
$$(1), (2), (3) \implies \text{(ii)}$$

hold for all $(a, b, c) \in K^3$? For which subfields K does the answer of (iii) make sense?

6. Let $I = \langle x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\rangle$.

(a) Show that the given generators of I are not a Gröbner basis for any lex order (i.e. for any of the 6 possible orderings of indeterminates: $x > y > z$, or $x > z > y$, or...)

(b) Find a monomial order for which the leading terms of the generators are coprime.

(c) Deduce that the generators are a Gröbner basis for the monomial order of (b).

7. For each of the following ideals $I \subset \mathbb{Q}[x, y, z]$, find the reduced Gröbner basis for the degrevlex and lex orders. Compare the number of basis elements and their complexity (e.g. total degree, maximal coefficients).

(a) $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\rangle$

(b) $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle$