MAST31005 Algebra II exercise 7 (13.03.2024)

1. Let $I \subset \mathbb{C}[x, y, z]$ be the ideal determined by the polynomial system

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ x^2 + 2y^2 &= 5 \\ xz &= 1. \end{aligned}$$

(a) Find bases for the elimination ideals I_1 and I_2.

(b) How many rational solutions does the system have?

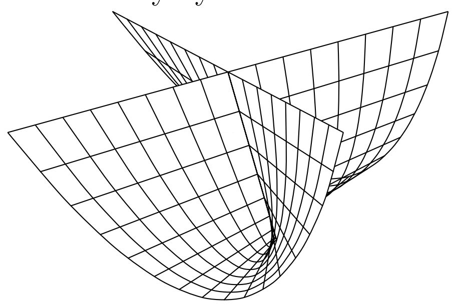
2. Call a monomial order $>$ on $K[x_1, \dots, x_n]$ of l-*elimination type* if any monomial involving any of $x_1, \dots, x_l$ is greater than all monomials in $K[x_{l+1}, \dots, x_n]$.

(a) Define a monomial order $>$ that is of l-elimination type and restricts to degrevlex on both $K[x_1, \dots, x_l]$ and $K[x_{l+1}, \dots, x_n]$.

(b) Let G be a Gröbner basis of an ideal $I \subset K[x_1, \dots, x_n]$ with respect to a monomial order of l-elimination type. Show that $G \cap K[x_{l+1}, \dots, x_n]$ is a Gröbner basis of the elimination ideal $I_l = I \cap K[x_{l+1}, \dots, x_n]$.

3. The Whitney umbrella $W \subset \mathbb{R}^3$ is given parametrically by

$$\begin{aligned} x &= uv, \\ y &= v, \\ z &= u^2. \end{aligned}$$



(a) Find the smallest variety $V \subset \mathbb{R}^3$ containing the Whitney umbrella W.

(b) Show that $W \neq V$.

(c) Find all the points $(x, y, z) \in W$ for which the parameters $u, v \in \mathbb{R}$ are not uniquely determined. Explain how this is seen in the above picture.

4. Let $J = \langle x^2 + y^2 - 1, y - 1 \rangle \subset \mathbb{R}[x, y]$.

(a) Draw the varieties $V(x^2 + y^2 - 1)$, $V(y - 1)$, and $V(J)$ in the plane $\mathbb{R}^2$.

(b) Find $f \in I(V(J))$ such that $f \notin J$.