MAST31005 Algebra II exercise 8 (20.03.2024)

1. Show that each variety $V \subset \mathbb{R}^n$ can be defined by a single equation, i.e., show that $V = V(p)$ for some $p \in \mathbb{R}[x_1, \ldots, x_n]$.

2. Let K be a field which is **not** algebraically closed. Let $p = a_n x^n + \ldots + a_0 \in K[x]$. Define the *homogenization of* p to be the homogeneous polynomial

$$p_{\text{hom}} = a_n x^n + a_{n-1} x^{n-1} y + a_{n-2} x^{n-2} y^2 + \ldots + a_0 y^n \in K[x, y].$$

(a) Show that p has a root in K if and only if $p_{\text{hom}}(a, b) = 0$ for some $0 \neq (a, b) \in K^2$.

(b) Show that there exists $q \in K[x, y]$ such that $V(q) = \{(0, 0)\}$.

(c) Show that for each $n \geq 1$, there exists a polynomial $q_n \in K[x_1, \ldots, x_n]$ such that $V(q_n) = \{(0, \ldots, 0)\}$.

(d) Let $W = V(p_1, \ldots, p_s) \subset K^n$ be a variety. Show that $W = V(f)$, where $f = q_n(p_1, \ldots, p_s)$. Compare this to Exercise 1.

3. Let K be an arbitrary field and let

$$S = \{p \in K[x_1, \ldots, x_n] : p(a) \neq 0 \ \forall a \in K^n\}$$

be the subset of polynomials with no zeros. Let $I \subset K[x_1, \ldots, x_n]$ be an ideal such that $I \cap S = \emptyset$. Show that $V(I) \neq \emptyset$. Hint: when K is not algebraically closed, use Exercise 2.

4. Define ideals $I, J \subset \mathbb{C}[x, y]$ such that $I \not\subset J$ and $J \not\subset I$, but $V(I) = V(J) \neq \emptyset$.

5. Show that $\langle x^2 + 1 \rangle \subset \mathbb{R}[x]$ is a radical ideal, but $V(x^2 + 1) = \emptyset$.

6. (a) Let $I \subset K[x_1, \ldots, x_n]$ be an ideal such that $\sqrt{I} = \langle p_1, p_2 \rangle$ with $p_1^{m_1} \in I$ and $p_2^{m_2} \in I$. Show that $q^{m_1 + m_2 - 1} \in I$ for all $q \in \sqrt{I}$. Hint: look at the proof in Lemma 9.6 that $p, q \in \sqrt{I} \implies p + q \in \sqrt{I}$.

(b) Let $I \subset K[x_1, \ldots, x_n]$ be any ideal. Show that there exists $M \in \mathbb{N}$ such that $q^M \in I$ for all $q \in \sqrt{I}$.

7. (a) Let $I = \langle x^3, y^3, xy(x + y) \rangle \subset K[x, y]$. Is $x + y \in \sqrt{I}$? If so, what is the smallest power such that $(x + y)^m \in I$?

(b) Let $I = \langle x + z, x^2 y, x - z^2 \rangle \subset K[x, y]$. Is $x^2 + 3xz \in \sqrt{I}$? If so, what is the smallest power such that $(x^2 + 3xz)^m \in I$?

8. (a) Show that $\sqrt{\langle x^n, y^m \rangle} = \langle x, y \rangle \subset K[x, y]$ for all $n, m \geq 1$.

(b) Let $p, q \in K[x, y]$ and $I = \langle p^2, q^3 \rangle$. Is it always true that $\sqrt{I} = \langle p, q \rangle$?