Definition 3.7

Let $A \in K^{nm}$, $B \in K^{pq}$ be matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

The Kronecker product (aka. tensor product) of A and B is the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix} \in K^{np \times mq}$$

Proposition 3.8

Let $K \hookrightarrow L$ and $\alpha, \beta \in L$ eigenvalues of matrices $A \in K^{n \times n}$ and $B \in K^{m \times m}$ respectively. Then

(i) $-\alpha$ is an eigenvalue of $-A$

(ii) $1/\alpha$ is an eigenvalue of A^{-1} if A invertible

(iii) $\alpha\beta$ is an eigenvalue of $A \otimes B$

(iv) $\alpha + \beta$ is an eigenvalue of $A \otimes I_m + I_n \otimes B$ where $I_n \in K^{n \times n}$, $I_m \in K^{m \times m}$ are identity matrices.

Proof

Let $u \in L^n$ and $v \in L^m$ be the corresponding eigenvectors

$$Au = \alpha u \quad \text{and} \quad Bv = \beta v.$$

(i) $(-A)u = -Au = -\alpha u$

$\Rightarrow$ u eigenvector of $-A$ with eigenvalue $-\alpha$

(ii) $A^{-1}u = \frac{1}{\alpha} A^{-1}(\alpha u) = \frac{1}{\alpha} A^{-1}Au = \frac{1}{\alpha} u$

$\Rightarrow$ u eigenvector of A^{-1} with eigenvalue $1/\alpha$

(iii) Consider the column vector

$$u \otimes v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} u_1 v \\ \vdots \\ u_n v \end{pmatrix} \in L^{nm}$$

By block-matrix multiplication

$$(A \otimes B)(u \otimes v) = \begin{pmatrix} a_{11} B(u_1 v) + \dots + a_{1n} B(u_n v) \\ \vdots \\ a_{n1} B(u_1 v) + \dots + a_{nn} B(u_n v) \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}u_1 + \dots + a_{1n}u_n) Bv \\ \vdots \\ (a_{n1}u_1 + \dots + a_{nn}u_n) Bv \end{pmatrix}$$

$$= (Au) \otimes (Bv)$$

$$= (\alpha u) \otimes (\beta v) = \alpha\beta (u \otimes v)$$

$\Rightarrow$ $u \otimes v$ eigenvector of $A \otimes B$ with eigenvalue $\alpha\beta$

(iv) $(A \otimes I + I \otimes B)(u \otimes v)$

$= (Au) \otimes (Iv) + (Iu) \otimes (Bv)$

$= \alpha (u \otimes v) + \beta (u \otimes v)$

$= (\alpha + \beta)(u \otimes v)$ $\square$

Example 3.9

Let $\alpha = i$, $\beta = \sqrt[3]{2}$. Their minimal polynomials over $\mathbb{Q}$ and corresponding companion matrices are

$$p_\alpha = 1 + 0 \cdot t + t^2$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$p_\beta = -2 + 0 \cdot t + 0 \cdot t^2 + t^3$$

$$B = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix} = \begin{pmatrix} & & & & & -2 \\ & & & -1 & 0 & \\ & & & 0 & -1 & \\ & & 2 & & & \\ 1 & 0 & & & & \\ 0 & 1 & & & & \end{pmatrix}$$

$$\det(tI_6 - A \otimes B) = t^6 + 4$$

$$(i \sqrt[3]{2})^6 = i^6 \cdot (\sqrt[3]{2})^6 = 4$$

$$A \otimes I_3 + I_2 \otimes B = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

$$\Rightarrow \det(tI_6 - A \otimes I_3 - I_2 \otimes B)$$

$$= t^6 + 3t^4 - 4t^3 + 3t^2 + 12t + 5$$

is a polynomial with root $i + \sqrt[3]{2}$.

(also happens to be its minimal polynomial)

3B CONSTRUCTIBLE NUMBERS

Constructible refers to ruler and compass constructions.

Operations:

(1) Given two points $z, w \in \mathbb{C}$, $z \neq w$ construct the (infinite) line $L(z,w)$

(2) Given a point $z \in \mathbb{C}$ and a radius $r > 0$ construct the circle $C(z,r)$

<u>Definition 3.10</u>

- Let P_n, $\mathcal{L}_n$, C_n, $n \in \mathbb{N}$, be the recursively defined sets of <u>n-constructible</u> points, lines, circles:

$$P_0 = \{0, 1\} \subset \mathbb{C}, \quad L_0 = \emptyset, \quad C_0 = \emptyset$$

$$\mathcal{L}_{n+1} = \{L(z,w) : z, w \in P_n\}$$

$$C_{n+1} = \{C(z, |w-u|), \ z, w, u \in P_n\}$$

$$P_{n+1} = \text{"intersections of distinct objects of } \mathcal{L}_{n+1} \cup C_{n+1}\text{"}$$

$$= \{z \in \mathbb{C} : \exists A, B \in \mathcal{L}_{n+1} \cup C_{n+1}, \ A \neq B, \ z \in A \cap B\}$$

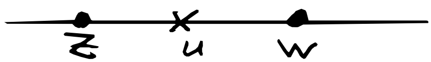
- An element $z \in \mathbb{C}$ is <u>constructible</u> if $z \in \bigcup_{n \in \mathbb{N}} P_n$.

Denote by $\mathcal{P}$ the set of constructible numbers.

The elementary line and circle constructions lead to increasingly complicated constructions, Search for "Euclid: The Game" to try a digital version

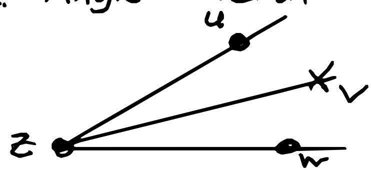
Some possible constructions:

1. Line Bisection



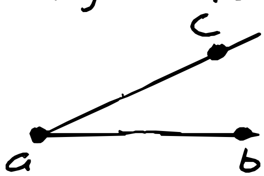
$z, w \rightsquigarrow u \in L(z,w)$ s.t. $|z-u| = |w-u|$

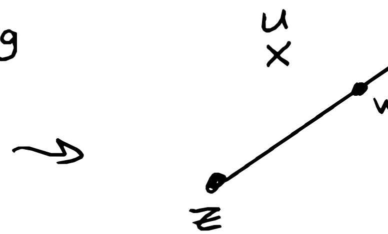
2. Angle bisection



$z, u, w \rightsquigarrow v$ s.t. $\measuredangle wzv = \frac{1}{2} \measuredangle wzu$

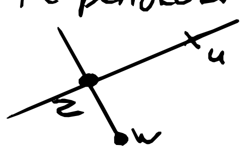
3. Angle repositioning





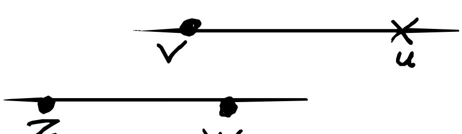
$a, b, c, z, w \rightsquigarrow u$ s.t. $\measuredangle bac = \measuredangle wzu$

4. Perpendicular line



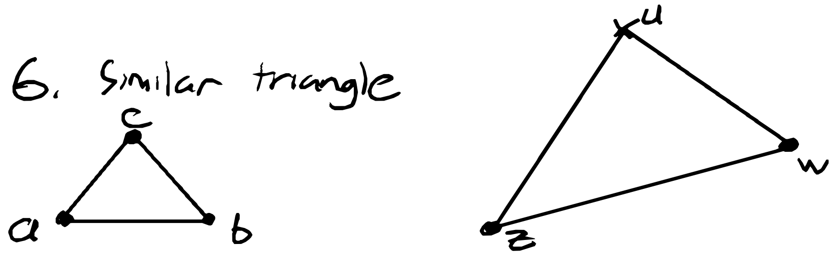
$z, w \rightsquigarrow u$ s.t. $\measuredangle wzu = \frac{\pi}{2}$

5. Parallel line



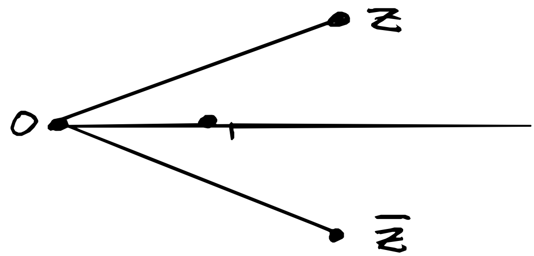
$z, w, v \rightsquigarrow u$ s.t. $L(u,v)$ parallel to $L(z,w)$

6. Similar triangle



$a, b, c, z, w \rightsquigarrow u$ s.t. triangle (z, w, u) similar to triangle (a, b, c)

7. Complex conjugate



Proposition 3.11

Suppose $z, w \in \mathbb{C}$ are constructible.
Then all of the following are constructible

(i) $z+w$

(ii) $-z$

(iii) zw

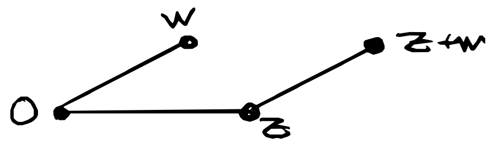
(iv) $1/z$

(v) $\pm\sqrt{z}$

Proof by picture

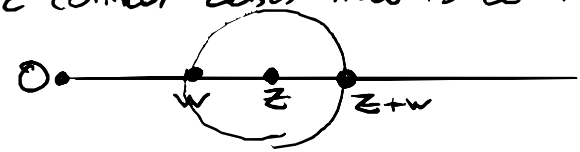
Using the previous constructions 1–7:

(i)



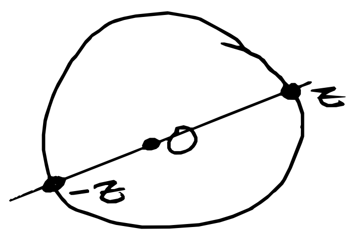
(parallel line)

Note colinear cases need to be handled separately:



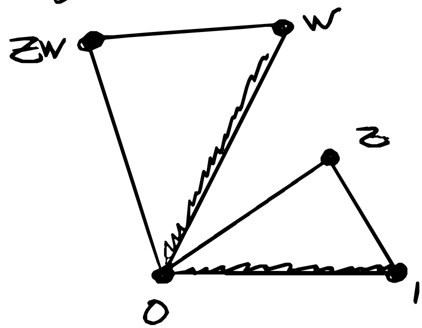
$C(z, |w-z|)$

(ii)

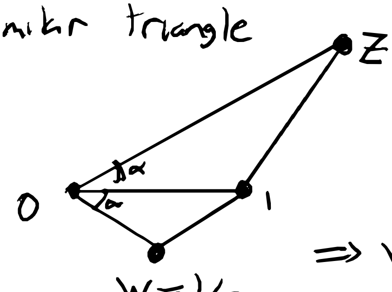


$C(0, |z|)$

(iii) similar triangle:



(iv) similar triangle

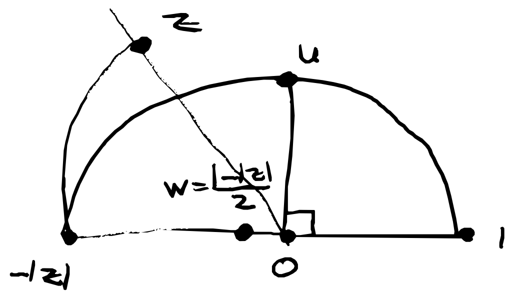


$$\frac{|w|}{1} = \frac{1}{|z|}$$

& w colinear with $\overline{z}$

$$\Rightarrow w = \frac{\overline{z}}{|\overline{z}|} \cdot |w| = \frac{\overline{z}}{|z|^2} = z^{-1}$$

(v)

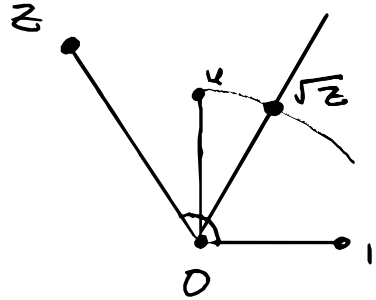


bisect Segment $-|z|$ to 1 $\rightsquigarrow w$

$C(w, |w-1|)$ & perpendicular line to $L(0,1)$ through 0

intersect at u.

Geometric mean theorem $\Rightarrow |u| = \sqrt{|-z| \cdot 1} = \sqrt{|z|}$



Bisect angle $\measuredangle 1 0 z$ & intersect $C(0, |u|)$

If $z = re^{i\theta}$, intersection is $\sqrt{r} \cdot e^{i\theta/2} = \sqrt{z}$ $\square$

Definition 3.12

A field K is *quadratically closed* if every $p \in K[t]$ with $\deg p = 2$ has a root.

Proposition 3.11 $\Rightarrow$ the set of constructible numbers is a quadratically closed field.

Theorem 3.13

The field of constructible numbers P is the quadratic closure of $\mathbb{Q}$, i.e., the union of all subfields $K_n \subset \mathbb{C}$ s.t. $\exists$ a chain of extensions

$$\mathbb{Q} = K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_n, \qquad [K_{i+1} : K_i] = 2.$$

Proof

Let $\mathbb{Q} = K_0 \hookrightarrow \cdots \hookrightarrow K_n$ be a chain of quadratic extensions. Since each $[K_{j+1} : K_j] = 2$, $K_{j+1} = K_j(\alpha_{j+1})$ with $\alpha_{j+1}^2 \in K_j$ (Exercise)

$\mathbb{Q}$ is constructible, so $K_1 = \mathbb{Q}(\alpha_1)$ is also constructible by Proposition 3.11, since $\alpha_1 = \pm\sqrt{\alpha_1^2}$, $\alpha_1^2 \in \mathbb{Q}$ constructible.

By induction each K_j is constructible.

$\Rightarrow$ P contains the quadratic closure.

For the converse, we will use the following:

<u>Lemma 3.14</u>

If $z \in \mathbb{C}$ is n-constructible, i.e. $z \in P_n$, then $\bar{z} \in P_n$.

<u>Proof of Lemma</u>

Consider the mirror image of the construction. $\square$