Proof of Thm 3.3 continued

Let F = quadratic closure of $\mathbb{Q}$.

Inductively, suppose $P_n \subset F$. Consider

P_{n+1} = intersections of lines & circles constructed from P_n.

Case 1: $L(z,w) \cap L(u,v)$, $z, w, u, v \in P_n$

$$L(z,w) = \{ z + \alpha(w-z) : \alpha \in \mathbb{R} \}$$
$$L(u,v) = \{ u + \beta(v-u) : \beta \in \mathbb{R} \}$$

$\rightsquigarrow$ linear system in variables α, β:

$$(w-z)\alpha - (v-u)\beta = z-u$$
$$(\bar{w}-\bar{z})\alpha - (\bar{v}-\bar{u})\beta = \bar{z}-\bar{u}$$

Abstractly, we have a system $Ax=y$,

$A \in F^{2\times 2}$, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $y = \begin{pmatrix} z-u \\ \bar{z}-\bar{u} \end{pmatrix} \in F^2$.

The solution is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = x = A^{-1} y \in F^2$$

$\Rightarrow \alpha \in F \Rightarrow z + \alpha(w-z) \in F$.

Case 2: $L(z_1, z_2) \cap C(z_3, |z_4 - z_5|)$, $z_1, \dots, z_5 \in P_n \subset F$.

$$z_1 + \alpha(z_2 - z_1) = z_3 + re^{i\theta}, \quad r = |z_4 - z_5|, \ \alpha, \theta \in \mathbb{R}.$$

$$\bar{z}_1 + \alpha(\bar{z}_2 - \bar{z}_1) = \bar{z}_3 + re^{-i\theta} \quad \text{(mirrored)}$$

Manipulate equations to obtain

$$\underset{\substack{\| \\ r^2 = (z_4 - z_5)(\bar{z}_4 - \bar{z}_5)}}{re^{i\theta}\cdot re^{-i\theta}} = (z_1 + \alpha(z_2 - z_1) - z_3)(\bar{z}_1 + \alpha(\bar{z}_2 - \bar{z}_1) - \bar{z}_3)$$

$\rightsquigarrow$ α root of a quadratic poly with coefficients in $P_n \subset F$.

$\Rightarrow$ α in F (why? compare Exercise 2.5)

Case 3: $w \in C(z_1, |z_2 - z_3|) \cap C(z_4, |z_5 - z_6|)$, $z_1, \dots, z_6 \in P_n \subset F$.

$$|w - z_1| = |z_2 - z_3| \ \& \ |w - z_4| = |z_5 - z_6|$$

Rewrite as a quadratic system in $w, \bar{w}$:

$$(w - z_1)(\bar{w} - \bar{z}_1) = (z_2 - z_3)(z_2 - \bar{z}_3) = r^2$$

$$(w - z_4)(\bar{w} - \bar{z}_4) = (z_5 - z_6)(z_5 - \bar{z}_6) = s^2$$

Solve for $\bar{w}$

$$\frac{r^2}{w - z_1} + \bar{z}_1 = \bar{w} = \frac{s^2}{w - z_4} + \bar{z}_4$$

expand denominators $\rightsquigarrow$ w root of quadratic poly with coefficients in $P_n \subset F$.

$\Rightarrow w \in F$ as in Case 2. $\square$

Corollary 3.15

If $\alpha \in P$, then $[\mathbb{Q}(\alpha):\mathbb{Q}] = 2^n$ for some $n \in \mathbb{N}$.

Proof

By Theorem 3.13, $\exists$ chain $\mathbb{Q} \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_m \ni \alpha$, $[K_{j+1}:K_j] = 2$. Then $\mathbb{Q} \hookrightarrow \mathbb{Q}(\alpha) \hookrightarrow K_m$.

By the Tower Law (Thm 1.12)

$$[\mathbb{Q}(\alpha):\mathbb{Q}] \mid [K_m:\mathbb{Q}] = 2^m \quad \square$$

Corollary 3.16 (classic impossible constructions)

A ruler and compass construction cannot

(i) duplicate the cube (double the volume)

(ii) trisect all angles

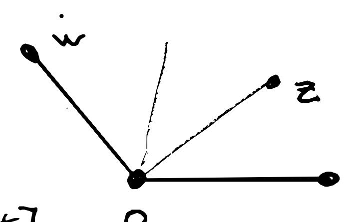
(iii) square the circle (square with area equal to a circle)

Proof sketch

(i) $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3 \neq 2^n$

(ii) $w = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z = e^{2\pi i/9}$



Minimal poly of w is $t^2+t+1 \in \mathbb{Q}[t]$

$\Rightarrow z$ root of $t^6+t^3+1 \in \mathbb{Q}[t]$, which is irreducible

$\Rightarrow [\mathbb{Q}(z):\mathbb{Q}] = 6 \neq 2^n$.

(iii) $[\mathbb{Q}(\pi):\mathbb{Q}] = \infty$. $\square$

3C ORIGAMI NUMBERS

Ruler and compass	Origami
Construct lines & circles	fold lines
and consider intersections	and consider intersections

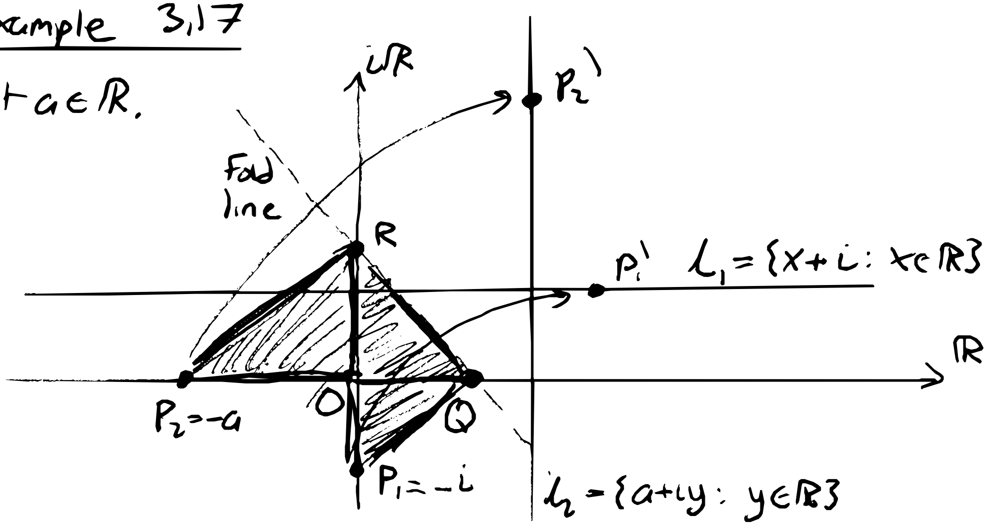
Mathematically origami can be reduced to:

Beloch's fold (1936):

Given points P_1, P_2 and lines ℓ_1, ℓ_2 simultaneously fold P_1 onto ℓ_1 and P_2 onto ℓ_2

Example 3.17

Let $a \in \mathbb{R}$.



A similar triangles argument shows

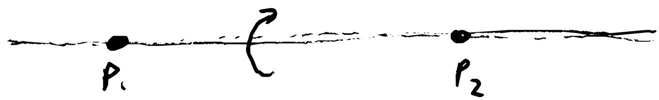
$$\Delta(P_2 O R) \simeq \Delta(R O Q) \simeq \Delta(Q O P_1)$$

$$\Rightarrow \frac{a}{|R|} = \frac{|R|}{|Q|} = \frac{|Q|}{1} \Rightarrow \begin{cases} |R| = |Q|^2 \\ |R|^2 = |Q|a \end{cases} \Rightarrow |Q| = \sqrt[3]{a}.$$

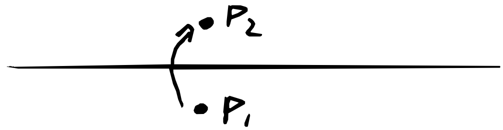
We also allow degenerate cases of Beloch's fold:

- no lines marked

  1) Fold a crease line connecting P_1 and P_2

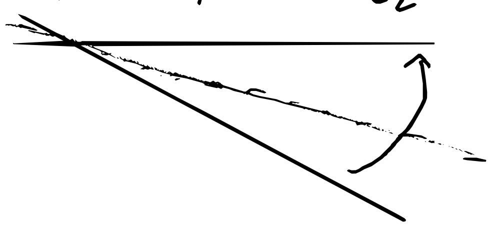


  2) fold P_1 onto P_2



- no points marked

  3) fold ℓ_1 onto ℓ_2

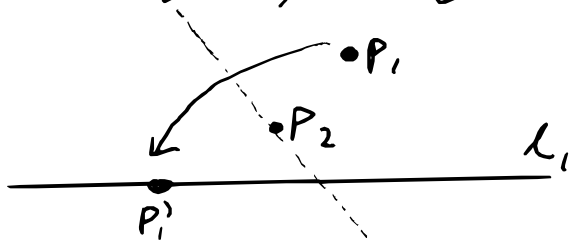


- only one point and line marked

  4) Fold perpendicular line through a point

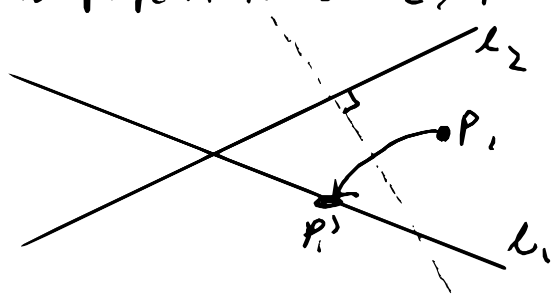
- only one line marked

5) Fold through P_2, placing P_1 onto l_1



- only one point marked

6) Fold perpendicular to l_2, placing P_1 onto l_1



Key feature: each of these folds has no degree of freedom (i.e. solutions are rigid)

Theorem (Lang 2003; Hull 2005)
Beloch's fold and folds 1–6 are the only possible folds with no degrees of freedom.

We omit the proof. Idea: consider possible ways to eliminate degrees of freedom specified by points and lines.

Definition 3.18

Let $\mathcal{O}_n$, $\mathcal{L}_n$ be the recursively defined sets of origami-n-constructible points and lines:

$$\mathcal{O}_0 = \{0, 1\} \subset \mathbb{C}, \quad \mathcal{L}_0 = \emptyset$$

$$\mathcal{L}_{n+1} = \{\text{crease lines formed using Beloch's fold and folds 1-6 from } \mathcal{O}_n \text{ and } \mathcal{L}_n\}$$

$$\mathcal{O}_{n+1} = \mathcal{O}_n \cup \{\text{intersections of } l_1, l_2 \in \mathcal{L}_{n+1}\}$$

The set of origami-numbers is $\mathcal{O} = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n \subset \mathbb{C}$.

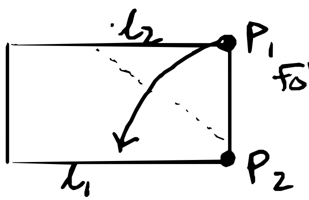
<u>Note:</u> In this mathematical model of origami, there is no "folded state".

In practical constructions folds of folded paper are allowed, effectively allowing multiple creases simultaneously.

This distinction does not change $\mathcal{O}$: reflections of points & lines across a line are origami-constructible.

Similarly if $P \in \mathcal{O}$ and a constructible fold takes P to P', then $P' \in \mathcal{O}$.

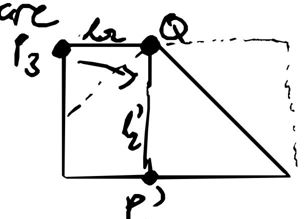
Practically, this means we can use folded lines and points as "virtual" lines and points, reducing the need for auxiliary constructions.



Fold-5: $P_1 \rightsquigarrow l_1$ through P_2

let Q intersection of l_2 and crease

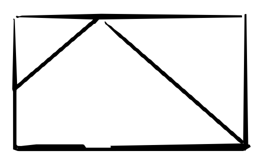
Compare



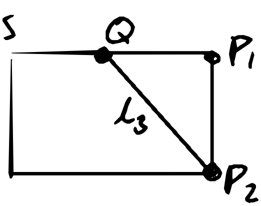
fold - 5 $\rightsquigarrow$

$P_3 \rightsquigarrow l_2'$

Fold through Q

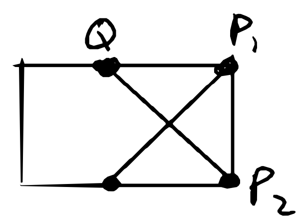


versus

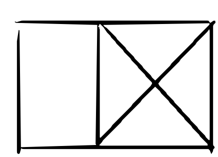


Fold-4 $\rightsquigarrow$

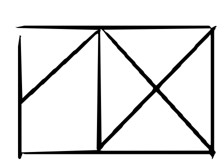
perpendicular to l_3 through P_1



Fold-1 $\rightsquigarrow$

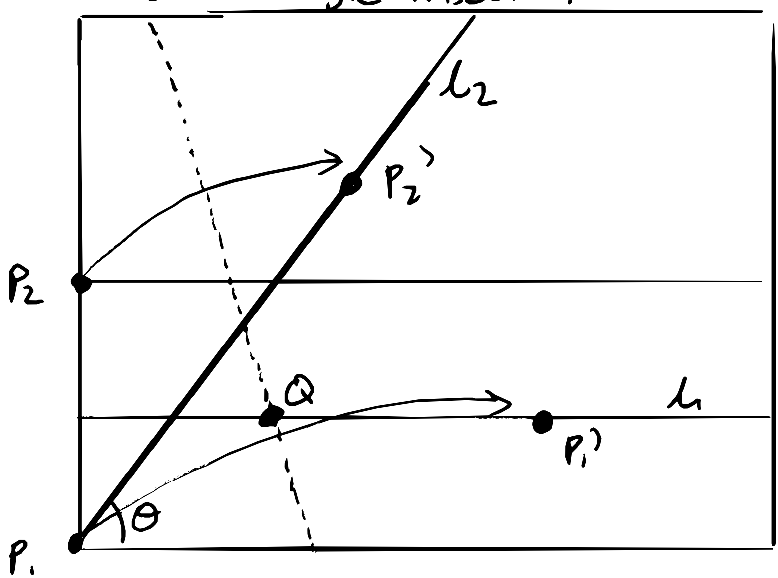


Fold-5 $\rightsquigarrow$



Example 3.19

Abe's angle trisection:



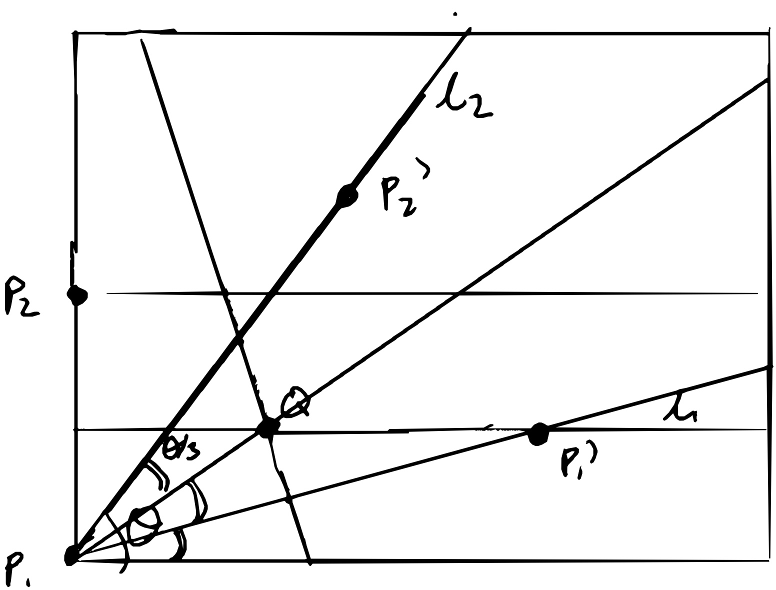
θ acute angle between bottom boundary of paper and ℓ_2.

ℓ_1 line $\frac{1}{4}$ up.

P_1 bottom left corner

P_2 halfway up left side

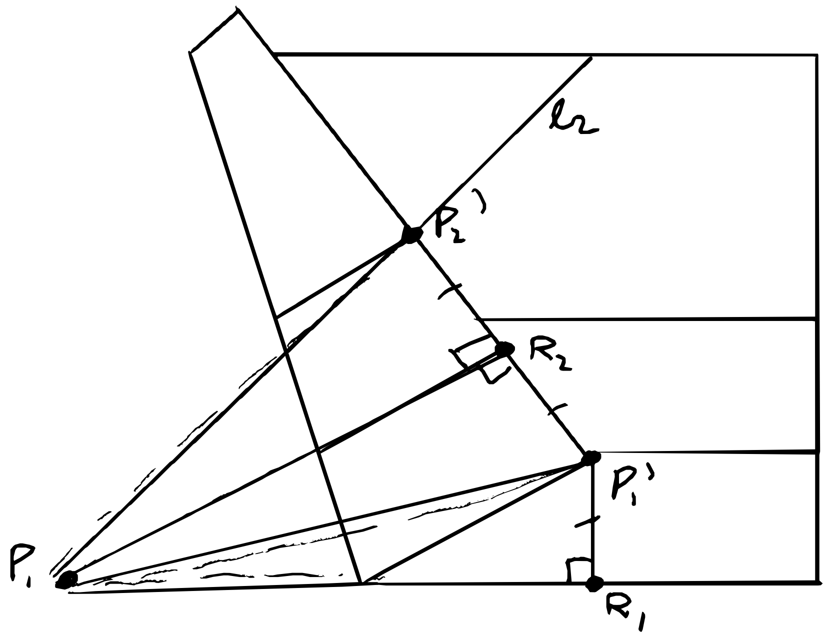
Fold $P_1 \rightsquigarrow \ell_1$, $P_2 \rightsquigarrow \ell_2$



Let Q intersection of fold crease and ℓ_1.

Folds through P_1 and Q and through P_1 and P_1' trisect θ

Proof of trisection:



By construction $|R_1P_1'| = |P_1'R_2| = |R_2P_2'|$

$\rightsquigarrow \quad \Delta(P_1R_1P_1') \simeq \Delta(P_1P_1'R_2) \simeq \Delta(P_1R_2P_2')$