x_1-decompositions for $p \in K[x_1, \dots, x_n]$

$$p = c \cdot x_1^N + r,$$

where

$c \in K[x_2, \dots, x_n]$, $c \neq 0$

$N \geq 0$

all terms of r have x_1-degree $< N$

If $p \in K[x_2, \dots, x_n]$, then

$p = c$, $N = 0$, $r = 0$

In Extension Theorem statement, we have

$$\bar{a} \notin V(c_1, \dots, c_s) \;\&\; \bar{a} \in V(I_1)$$

$$\Updownarrow$$

$$\bar{a} \notin V(c_i : N_i > 0, 1 \leq i \leq s) \;\&\; \bar{a} \in V(I_1)$$

since

$\bar{a} \in V(I_1) \Rightarrow$ if $N_i = 0$, then $p_i \in I_1$ so

$$p_i(\bar{a}) = c_i(\bar{a}) = 0$$

Theorem 8.8.

Let k be algebraically closed and $I \subset k[x_1, \ldots, x_n]$ an ideal.
Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis of I in lex order.
Let $g_j = c_j x^{N_j} + r_j$ be the x_1-decompositions.
Let $\bar{a} = (a_2, \ldots, a_n) \in V(I_1)$ be a partial solution such that $\bar{a} \notin V(c_1, \ldots, c_t)$. Then

$$\{f(x_1, \bar{a}) : f \in I\} = \langle g(x_1, a) \rangle \subset k[x_1]$$

where $\bar{g} \in G$ is an element with minimal $\deg(\bar{g}, x_1)$ such that $\bar{c} := c_{\bar{g}}$ satisfies $\bar{c}(\bar{a}) \neq 0$
Moreover, $\deg \bar{g}(x_1 \bar{a}) > 0$.

Proof

Let $\bar{g} \in G$ be such that $\bar{c}(\bar{a}) \neq 0$ and $\deg(g, x_1)$ minimal.

(i) degree of $\bar{g}(x_1, \bar{a}) \in k[x_1]$ is nonzero:

we have

$$\bar{g}(x_1, \bar{a}) = \bar{c}(\bar{a}) x_1^{\bar{N}} + \bar{r}(\bar{a})$$

so $\deg(\bar{g}(x_1 \bar{a})) = 0$ and $\bar{c}(\bar{a}) \neq 0$ would imply $\bar{N} = 0$,

$\Rightarrow \bar{g} = \bar{c} \in k[x_2, \ldots, x_n] \Rightarrow \bar{g} \in I_1$.

Then $\bar{a} \in V(I_1)$ implies $\bar{c}(\bar{a}) = \bar{g}(\bar{a}) = 0$ $\lightning$

Let $J := \{f(x_1, \bar{a}) ; f \in I\} \subset k[x_1]$

(ii) J is an ideal generated by $\{g_1(x_1, \bar{a}), \ldots, g_t(x_1, \bar{a})\}$

Consider the ring homomorphism

$\varphi : k[x_1, \ldots, x_n] \to k[x_1] : \quad \varphi(f) := f(x_1, \bar{a})$

ring homomorphism $\Rightarrow$ $J = \varphi(I)$ is an ideal and $\varphi(G)$ is a basis of J : $\varphi(\sum q_i g_i) = \sum \varphi(q_i)\varphi(g_i)$.

It suffices to show $g_j(x_1, \bar{a}) \in \langle \bar{g}(x_1, \bar{a}) \rangle$, $j = 1, \dots, t$.
The theorem will follow since G is a basis of I.
We will show this by induction on $\deg(g_j, x_1)$.

(iii) $\underline{\deg(g_j, x_1) < \deg(\bar{g}, x_1) \Rightarrow g_j(x_1, \bar{a}) = 0.}$

$\deg(\bar{g}, x_1)$ is minimal by construction
$\Rightarrow$ if $\deg(g_j, x_1) < \deg(\bar{g}, x_1)$ then $c_j(\bar{a}) = 0$
Arguing for a contradiction suppose some such $g_j(x_1, \bar{a}) \neq 0$.
Let g_B be such that $g_B(x_1, \bar{a}) \neq 0$ and
$\delta = \deg(g_B, x_1) - \deg(g_B(x_1, \bar{a}))$ is minimal.
(so if g_j satisfies $\deg(g_j, x_1) < \deg(\bar{g}, x_1)$ & $g_j(x_1, \bar{a}) \neq 0$, then $\deg(g_j, x_1) - \deg(g_j(x_1, \bar{a})) \geq \delta$)
Consider

$$S = \bar{c} \cdot x_1^{\bar{N} - N_B} \cdot g_B - c_B \cdot \bar{g}$$

" a S-polynomial of g_B & $\bar{g}$ in $(k[x_2, \dots, x_n])[x_1]$ "

To find a contradiction, we compute $\deg(S(x_1, \bar{a})) < \bar{N}$ in 2 ways.

Method 1: Evaluate at $\bar{a}$:

$$S(x_1,\bar{a}) = \bar{c}(\bar{a})\, x_1^{\bar{N}-N_B}\, g_B(x_1,\bar{a}) - \boxed{c_B(\bar{a})}^{=0}\, \bar{g}(x_1,\bar{a})$$
$$= \bar{c}(\bar{a})\, x_1^{\bar{N}-N_B}\, g_B(x_1,\bar{a})$$
$$\Rightarrow \deg S(x_1,\bar{a}) = \bar{N} - N_B + \deg(g_B(x_1,\bar{a}))$$
$$= \bar{N} - N_B + N_B - \delta = \bar{N} - \delta \qquad (1)$$

Method 2: standard representation

$$S = \sum_{j=1}^{t} q_j g_j \qquad (\text{exists since } S \in I)$$

Lemma 8.7(1) implies

$$\bar{N} > \deg(S, x_1) \geq \deg(q_j g_j, x_1) \geq \deg(q_j, x_1) + \deg(g_j, x_1)$$

whenever $q_j g_j \neq 0$. In particular each g_j appearing in the sum has $\deg(g_j, x_1) < \bar{N} \Rightarrow c_j(\bar{a}) = 0$.

By minimality of δ, we have $g_j(x_1,\bar{a}) = 0$ or

$$\deg g_j(x_1,\bar{a}) \leq \deg(g_j, x_1) - \delta$$

Hence

$$\deg S(x_1,\bar{a}) \leq \max\left(\deg q_j(x_1,\bar{a}) + \deg g_j(x_1,\bar{a})\right)$$
$$\leq \max\left(\deg(q_j, x_1) + \deg(g_j, x_1)\right) - \delta$$
$$\leq \deg(S, x_1) - \delta < \bar{N} - \delta \qquad (2)$$

(1) & (2) are contradictory so g_B with $g_B(x_1,\bar{a}) \neq 0$ cannot exist.

(iv) $\underline{g_j(x_1\bar a) \in \langle \bar g(x_1\bar a)\rangle \text{ for } \deg(g_j, x_1) \geq \bar N}$.

By induction suppose $g_j(x_1\bar a) \in \langle \bar g(x_1\bar a)\rangle$ whenever $\deg(g_j, x_1) < N$ for some $N \geq \bar N$

Let g_j such that $\deg(g_j, x_1) = N$ and consider

$$S = \bar c \cdot g_j - c_j \cdot x_1^{N-\bar N} \bar g$$

"S-polynomial of g_j & $\bar g$ in $(K[x_2, \ldots, x_n])[x_1]$"

Take a standard representation $S = \sum_{l=1}^{t} q_l g_l$

$\Rightarrow N > \deg(S, x_1) \geq \deg(q_l, x_1) + \deg(g_l, x_1)$

whenever $q_l \neq 0$ as in (iii).

By the inductive assumption we deduce

$$\bar c(\bar a) \cdot g_j(x_1\bar a) = c_j(x_1\bar a) \cdot x_1^{N-\bar N} \bar g(x_1\bar a) + \sum_{l=1}^{t} q_l(x_1\bar a) g_l(x_1\bar a)$$

$$\in \langle \bar g(x_1\bar a)\rangle$$

So also $g_j(x_1\bar a) \in \langle \bar g(x_1\bar a)\rangle$ since $\bar c(\bar a) \neq 0$. $\square$

Theorem 8.4 (Extension Theorem) [restatement]

Let K be an algebraically closed field and $I = \langle p_1, \dots, p_s \rangle \subset K[x_1, \dots, x_n]$.

$$p_i = c_i \cdot x_1^{N_i} + r_i \quad \text{the } x_1\text{-decompositions}$$

If $(a_2, \dots, a_n) \notin V(c_1, \dots, c_s)$

then $\exists a_1 \in K$ s.t. $(a_1, a_2, \dots, a_n) \in V(I)$.

Proof

Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis of I in lex.

Let $\bar{a} = (a_2, \dots, a_n)$. By the assumption on $\bar{a}$, we have

$c_i(\bar{a}) \neq 0$ for some i.

Consider a standard representation $p_i = \sum_{j=1}^{t} q_j g_j$.

By Lemma 8.7(ii) we have

$$0 \neq c_i(\bar{a}) = \sum_{\deg(q_j g_j, x_1) = N_i} c_{q_j}(\bar{a}) \cdot c_{g_j}(\bar{a})$$

so for some $g_j \in G$ we have $c_{g_j}(\bar{a}) \neq 0$.

Theorem 8.8 $\Rightarrow$ $\exists \bar{g} \in G$ such that $\deg \bar{g}(x_1, \bar{a}) > 0$ and

$$\{ f(x_1, \bar{a}) : f \in I \} = \langle \bar{g}(x_1, \bar{a}) \rangle \subset K[x_1]$$

K algebraically closed $\Rightarrow$ $\exists a_1 \in K$ s.t. $\bar{g}(a_1, \bar{a}) = 0$.

Then also $f(a_1, \bar{a}) = 0 \ \forall f \in I$

$\Rightarrow (a_1, \bar{a}) \in V(I)$ $\square$

Definition 8.9

In K^n, we will denote by $\pi_\ell : K^n \longrightarrow K^{n-\ell}$ the projection $\pi_\ell(a_1, \ldots, a_n) = (a_{\ell+1}, \ldots, a_n)$.

Lemma 8.10

Let $V = V(p_1, \ldots, p_s) \subset K^n$, $I = \langle p_1, \ldots, p_s \rangle \subset k[x_1, \ldots, x_n]$.
Then $\pi_\ell(V) \subset V(I_\ell)$

Proof

Let $(a_1, \ldots, a_n) \in V$ and $f \in I_\ell \subset k[x_{\ell+1}, \ldots, x_n]$

$p_1(a_1, \ldots, a_n) = \cdots = p_s(a_1, \ldots, a_n) = 0$

$\Rightarrow f(a_1, \ldots, a_n) = f(a_{\ell+1}, \ldots, a_n) = 0$

viewed as an element of $k[x_1, \ldots, x_n]$ viewed as an element of $k[x_{\ell+1}, \ldots, x_n]$

Hence $\pi_\ell(a) \in V(I_\ell)$. $\square$

By Lemma 8.10, we may write

$$\pi_\ell(V) = \{ (a_{\ell+1}, \ldots, a_n) \in V(I_\ell) : \exists a_1, \ldots, a_\ell \in K \quad (a_1, \ldots, a_n) \in V \}$$

so $\pi_\ell(V)$ consists of <u>exactly</u> the partial solutions that extend to complete solutions.

Example 8.11

In general $\pi_\ell(V)$ is not a variety!

For instance in Example 8.3

$$V = V(xy-1, xz-1) \in \mathbb{R}^3$$

$$\pi_1(V) = \{(a,a) \in \mathbb{R}^2 : a \neq 0\}$$

Theorem 8.12 (Geometric Extension Theorem)

Let k be an algebraically closed field and $V = V(p_1, \dots, p_s) \subset k^n$, $I = \langle p_1, \dots, p_s \rangle \subset k[x_1, \dots, x_n]$,

$$p_i = c_i \, x_i^{N_i} + r_i \quad \text{the } x_1\text{-decompositions}$$

Then

$$V(I_1) = \pi_1(V) \cup \left(V(c_1, \dots, c_s) \cap V(I_1) \right)$$

Proof

"$\supset$" Follows from Lemma 8.10.

"$\subset$" Let $\bar{a} \in V(I_1)$. If $\bar{a} \notin V(c_1, \dots, c_s)$ then

Extension Theorem $\Rightarrow \exists a_1$ s.t. $(a_1, \bar{a}) \in V$

$$\Rightarrow \bar{a} = \pi_1(a_1, \bar{a}) \in \pi_1(V) \quad \square$$

Example 8.13

$V(c_1, \dots, c_s)$ may hide everything:

$$p_1 = (y-z)x^2 + xy - 1$$
$$p_2 = (y-z)x^2 + xz - 1$$

Then $c_1 = c_2 = y - z$ but

$I = \langle p_1, p_2 \rangle = \langle xz-1, y-z \rangle$

$I_1 = \langle y - z \rangle$

so the Geometric Extension Theorem only states

$$V(I_1) = \pi_1(V) \cup (V(y-z) \cap V(I_1))$$
$$= \pi_1(V) \cup V(I_1)$$

which tells us nothing about $\pi_1(V)$.

Later in the course we will able to make a more precise statement relating $\pi_\ell(V)$ and $V(I_\ell)$:

Closure Theorem:

- $V(I_\ell)$ is the smallest variety containg $\pi_\ell(V)$ (Zariski closure)
- If $V \neq \emptyset$, then $\exists$ lower dimensional variety W s.t. $V(I_\ell) \setminus W \subset \pi_\ell(V) \subset V(I_\ell)$