Theorem 9.29

Let $I, J < K[x_1, \dots, x_n]$ ideals. Then

$$I \cap J = (\underbrace{tI}_{} + \underbrace{(1-t)J}_{}) \cap K[x_1, \dots, x_n]$$

ideals in $K[x_1, \dots, x_n, t]$

Definition 9.30

Let $I < K[x_1, \dots, x_n]$ be an ideal and $f \in K[t]$.
Then $f(t) I < K[x_1, \dots, x_n, t]$ is the ideal

$$f(t) I := \langle f(t) \underbrace{h(x_1, \dots, x_n)}_{=: h(x)} : h \in I \rangle$$

For the proof of Theorem 9.29, we will need:

Lemma 9.31

Let $I < K[x_1, \dots, x_n]$ ideal and $f \in K[t]$.

(i) $I = \langle p_1, \dots, p_s \rangle \Rightarrow f(t) I = \langle f(t) p_1(x), \dots, f(t) p_s(x) \rangle$.

(ii) If $g = g(x, t) \in f(t) I$, then $g(x, a) \in I \ \forall a \in K$.

Proof

(i) Elements of $f(t) I$ are sums of polynomials

$$qfh = q(x, t) \cdot f(t) \cdot h(x),$$

with $q \in K[x_1, \dots, x_n, t]$, $h \in I < K[x_1, \dots, x_n]$.

Since $h \in I$, we have

$$h = \sum_{i=1}^{s} h_i \cdot p_i, \qquad h_i \in K[x_1, \dots, x_n]$$

Then

$$qfh = qf(\sum h_i p_i) = \sum_{i=1}^{s} (qh_i)(fp_i) = \sum_{i=1}^{s} \tilde{q}_i \cdot (fp_i)$$
$$\in \langle fp_1, \dots, fp_s \rangle \subset K[x_1, \dots, x_n, t]$$

so $f(t) I \subset \langle fp_1, \dots, fp_s \rangle$.

Conversely since $p_i \in I$, we have $fp_i \in f(t) I$,

so $\langle fp_1, \dots, fp_s \rangle \subset f(t) I$.

(ii) Let $g = g(x,t) \in f(t) I$. By (i), we have

$$g(x,t) = \sum q_i(x,t) f(t) p_i(x)$$
$$\Rightarrow g(x,a) = \sum \underbrace{q_i(x,a) f(a)}_{\in K[x_1, \dots, x_n]} p_i(x) \in I \qquad \square$$

Proof of Theorem 9.29

"$\subset$" Let $f \in I \cap J \subset K[x_1, \dots, x_n]$. Then

$$\left.\begin{array}{l} f \in I \Rightarrow tf \in tI \\ f \in J \Rightarrow (1-t) f \in (1-t) J \end{array}\right\} \Rightarrow \begin{array}{l} f = tf + (1-t) f \\ \in tI + (1-t) J \end{array}$$

"$\supset$" Let $f \in (tI + (1-t)J) \cap K[x_1, \dots, x_n]$

Then $f = g(x,t) + h(x,t)$, $g \in tI$, $h \in (1-t) J$.

Since $tI = \langle tp(x) : p \in I \rangle$, we observe that

$$g(x,t) \in tI \Rightarrow g(x,0) = 0$$

Similarly $h(x,t) \in (1-t)J \Rightarrow h(x,1) = 0$

By Lemma 9.31, we get

$$f(x,0) = h(x,0) \in J \text{ and } f(x,1) = g(x,1) \in I$$

$$\Rightarrow f = f(x) = f(x,0) = f(x,1) \in I \cap J \qquad \square$$

Example 9.32

Let $I = \langle x^2y \rangle$ and $J = \langle xy^2 \rangle$

Then

$$tI + (1-t)J = \langle tx^2y,\ xy^2 - txy^2 \rangle$$

Computing in lex order, we find

$$S(tx^2y, -txy^2 + xy^2) = y \cdot tx^2y + x(-txy^2 + xy^2)$$
$$= x^2y^2$$

$$S(tx^2y, x^2y^2) = y \cdot tx^2y - t \cdot x^2y^2 = 0$$

$$S(-txy^2 + xy^2, x^2y^2) = x(-txy^2 + xy^2) + t \cdot x^2y^2 = x^2y^2$$

so we get the Gröbner basis

$$tx^2y,\ -txy^2 + xy^2,\ x^2y^2$$

so

$$I \cap J = \left(tI + (1-t)J\right) \cap k[x,y] = \langle x^2y^2 \rangle$$

Theorem 9.33

Let $I_1, \ldots, I_m \subset K[x_1, \ldots, x_n]$ ideals. Let

$$J = t_1 I_1 + \cdots + t_m I_m + \langle 1 - t_1 - \cdots - t_m \rangle$$
$$\subset K[x_1, \ldots, x_n, t_1, \ldots, t_m],$$

where each $t_j I_j = \langle t_j p : p \in I_j \rangle \subset K[x_1, \ldots, x_n, t_1, \ldots, t_m]$

Then $I_1 \cap \cdots \cap I_m = J \cap K[x_1, \ldots, x_n]$

Proof

"$\subset$" Let $f \in I_1 \cap \cdots \cap I_m \subset K[x_1, \ldots, x_n]$. Then

$$f = t_1 f + \cdots + t_m f + (1 - t_1 - \cdots - t_m) f \in J$$

"$\supset$" Let $f \in J \cap K[x_1, \ldots, x_n]$. Then

$$f = g_1(x,t) + \cdots + g_m(x,t) + h(x,t) \cdot (1 - t_1 - \cdots - t_m)$$

with $g_i \in t_i I_i$.

As in the proof of Theorem 9.29, we observe

$$g_i \in t_i I_i \Rightarrow g_i(x, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m) = 0$$

Evaluating at $(t_1, \ldots, t_m) = (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots$, we get

$$f(x, 1, 0, \ldots, 0) = g_1(x, 1, 0, \ldots, 0) \in I_1$$
$$f(x, 0, 1, \ldots, 0) = g_2(x, 0, 1, 0, \ldots, 0) \in I_2$$
$$\vdots$$
$$f(x, 0, \ldots, 0, 1) = g_m(x, 0, \ldots, 0, 1) \in I_m$$

So $f \in I_1 \cap \cdots \cap I_m$ $\square$

Lemma 9.34

Let $I, J \subset k[x_1, \ldots, x_n]$ ideals. Then

$$IJ \subset I \cap J$$

Proof Suffices to consider

$$pq \in IJ \text{ with } p \in I \text{ and } q \in J.$$

Then
$$\left.\begin{array}{l} p \in I \Rightarrow pq \in I \\ p \in J \Rightarrow pq \in J \end{array}\right\} \Rightarrow pq \in I \cap J$$

$\square$

Theorem 9.34

Let $I, J \subset k[x_1, \ldots, x_n]$ ideals. Then

$$V(I \cap J) = V(I) \cup V(J)$$

Proof

"$\supset$" $I \cap J \subset I \Rightarrow V(I) \subset V(I \cap J)$

$I \cap J \subset J \Rightarrow V(J) \subset V(I \cap J)$

"$\subset$" $IJ \subset I \cap J \Rightarrow V(I \cap J) \subset V(IJ) = V(I) \cup V(J)$ $\square$

Proposition 9.35

Let $I, J \subset k[x_1, \ldots, x_n]$ ideals. Then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

Proof

"$\subset$" $f \in \sqrt{I \cap J} \Rightarrow f^n \in I \cap J \Rightarrow f \in \sqrt{I}$ and $f \in \sqrt{J}$

"$\supset$" $f \in \sqrt{I}$ and $f \in \sqrt{J} \Rightarrow f^m \in I$ and $f^l \in J$

$\Rightarrow f^{m+l} = f^m f^l \in I \cap J \Rightarrow f \in \sqrt{I \cap J}$ $\square$

ZARISKI CLOSURE

Proposition 9.36

Let $S \subset k^n$ be an arbitrary set.
Then the variety $V(I(S))$ is the smallest variety containing S, where
$$I(S) = \{p \in k[x_1, \ldots, x_n] : p(a) = 0 \ \forall a \in S\}$$
is defined exactly as $I(W)$ for a variety $W \subset k^n$.

Proof

Let $W \subset k^n$ be a variety such that $S \subset W$.
We need to show $V(I(S)) \subset W$.
By the ideal-variety correspondence, we have
$$W = V(I(W)),$$
So since both V and I are inclusion reversing,
$$S \subset W \Rightarrow I(S) \supset I(W) \Rightarrow V(I(S)) \subset V(I(W)) = W \quad \square$$

Definition 9.37

- Let $S \subset k^n$ be a subset. The Zariski closure of S is
$$\overline{S} = V(I(S))$$
- A subset $S \subset V$ of a variety V is Zariski dense in V if $V = \overline{S}$.

Lemma 9.38

Let $S, T \subset k^n$ subsets. Then

(i) $I(\overline{S}) = I(S)$

(ii) $S \subset T \Rightarrow \overline{S} \subset \overline{T}$

(iii) $\overline{S \cup T} = \overline{S} \cup \overline{T}$

Proof

(i) $S \subset \overline{S} \Rightarrow I(S) \supset I(\overline{S})$.

Conversely, if $p \in I(S)$, then $S \subset V(p)$.

Zariski closure definition $\Rightarrow \overline{S} \subset V(p) \Rightarrow p \in I(\overline{S})$

(ii) $S \subset T \subset \overline{T} \Rightarrow \overline{S} \subset \overline{T}$ since $\overline{T}$ is a variety.

(iii) $\overline{S} \cup \overline{T}$ is a variety containing $S \cup T$.

Let $S \cup T \subset W$ for some variety W. Then

$$\left.\begin{array}{l} S \subset W \Rightarrow \overline{S} \subset W \\ T \subset W \Rightarrow \overline{T} \subset W \end{array}\right\} \Rightarrow \overline{S} \cup \overline{T} \subset W$$

so $\overline{S \cup T} = \overline{S} \cup \overline{T}$. $\square$

Theorem 9.39 (Closure Theorem, part one)

Let K be algebraically closed.

Let $V = V(p_1, \ldots, p_s) \subset K^n$ and $\pi_\ell : K^n \to K^{n-\ell}$ the projection $\pi_\ell(x_1, \ldots, x_n) = (x_{\ell+1}, \ldots, x_n)$.

Let $I_\ell = \langle p_1, \ldots, p_s \rangle \cap K[x_{\ell+1}, \ldots, x_n]$ the ℓ-th elimination ideal.

Then $V(I_\ell)$ is the Zariski closure of $\pi_\ell(V)$

Proof

In Lemma 8.10, we showed $\pi_\ell(V) \subset V(I_\ell)$,

so it suffices to show $V(I_\ell) \subset V(I(\pi_\ell(V))) = \overline{\pi_\ell(V)}$

Let $p \in I(\pi_\ell(V)) \subset K[x_{\ell+1}, \ldots, x_n]$, so

$$p(a_{\ell+1}, \ldots, a_n) = 0 \quad \forall (a_{\ell+1}, \ldots, a_n) \in \pi_\ell(V)$$

$$\Rightarrow p(a_1, \ldots, a_n) = 0 \quad \forall (a_1, \ldots, a_n) \in V$$

viewed as an element of $K[x_1, \ldots, x_n]$.

By Hilbert's Nullstellensatz

$$p \in I(V) = I(V(\langle p_1, \ldots, p_s \rangle)) \Rightarrow p^m \in \langle p_1, \ldots, p_s \rangle$$

for some $m \in \mathbb{N}$. Then $p^m \in I_\ell$ so $p \in \sqrt{I_\ell}$.

Hence we have

$$\sqrt{I_\ell} \supset I(\pi_\ell(V)) \Rightarrow V(I_\ell) = V(\sqrt{I_\ell}) \subset V(I(\pi_\ell(V))) = \overline{\pi_\ell(V)}$$

$\square$