# IDEAL QUOTIENTS

## Definition 9.40

If $I, J \subset k[x_1, \ldots, x_n]$ ideals, then the ideal quotient of $I$ by $J$ is

$$I : J = \{ p \in k[x_1, \ldots, x_n] : pq \in I \ \forall q \in J \}$$

## Example 9.41

Consider $\langle xz, yz \rangle : \langle z \rangle$.

$q \in \langle z \rangle \iff q = h \cdot z, \quad h \in k[x_1, \ldots, x_n]$.

Hence $pq \in \langle xz, yz \rangle \ \forall q \in \langle z \rangle$

$$\begin{aligned}
&\iff p \cdot z \in \langle xz, yz \rangle \\
&\iff pz = f \cdot xz + g \cdot yz \\
&\iff p = f \cdot x + g \cdot y \\
&\iff p \in \langle x, y \rangle
\end{aligned}$$

So we have

$$\langle xz, yz \rangle : \langle z \rangle = \langle x, y \rangle.$$

## Proposition 9.42

If $I, J \subset k[x_1, \ldots, x_n]$ ideals,

then $I:J$ is an ideal and $I \subset I:J$.

## Proof

$I \subset I:J$

If $p \in I$ then $pq \in I$ $\forall q \in k[x_1, \ldots, x_n] \supset J$.

$I:J$ is an ideal:

- $0 \cdot q = 0 \in I$ $\forall q \in J \Rightarrow 0 \in I:J$
- $p_1, p_2 \in I:J \Rightarrow (p_1 + p_2)q = p_1 q + p_2 q \in I$ $\forall q \in J$
- $p \in I:J$, $h \in k[x_1, \ldots, x_n]$
  $\Rightarrow (hp)q = h(pq) \in I$ $\forall q \in J$ $\square$

## Proposition 9.43

If $I, J \subset k[x_1, \ldots, x_n]$ ideals, then

(i) $V(I) = V(I+J) \cup V(I:J)$
$= \big(V(I) \cap V(J)\big) \cup V(I:J)$ and

(ii) $\overline{V(I) \setminus V(J)} \subset V(I:J)$

If $V, W \subset k^n$ varieties, then

(iii) $V = (V \cap W) \cup \left(\overline{V \setminus W}\right)$

Proof

(iii) $V$ is a variety containing the set $V \setminus W$. Hence its Zariski closure satisfies $\overline{V \setminus W} \subset V$. so

$$V = (V \cap W) \cup (V \setminus W)$$
$$\subset (V \cap W) \cup (\overline{V \setminus W})$$
$$\subset V$$

(ii) The Zariski closure $\overline{V(I) \setminus V(J)}$ is

$$\overline{V(I) \setminus V(J)} = V(I(V(I) \setminus V(J)))$$

So $\overline{V(I) \setminus V(J)} \subset V(I : J)$ follows if we show

$$I : J \subset I(V(I) \setminus V(J))$$

Let $p \in I : J$ and $a \in V(I) \setminus V(J)$.
Then

$$p \in I : J \Rightarrow \forall q \in J \quad pq \in I$$
$$a \in V(I) \Rightarrow \forall q \in J \quad p(a) q(a) = 0$$
$$a \notin V(J) \Rightarrow \exists q \in J \quad q(a) \neq 0 \Rightarrow p(a) = 0$$
$$\Rightarrow p \in I(V(I) \setminus V(J)).$$

(i) By (iii) for $V = V(I)$ and $W = V(J)$, we get

$$V(I) = (V(I) \cap V(J)) \cup (\overline{V(I) \setminus V(J)})$$
$$\subset V(I+J) \cup V(I : J)$$

using (ii).

$\square$

## Example 9.44

In general $V(I:J) \neq \overline{V(I) \setminus V(J)}$

Let $I = \langle x^2(y-1) \rangle$, $J = \langle x \rangle$ in $\mathbb{C}[x,y]$.

Then

$$V(I) = V(x) \cup V(y-1) = V(J) \cup V(y-1),$$

with $\overline{V(I) \setminus V(J)} = V(y-1)$

but

$$\begin{aligned} p \in I:J &\iff p \cdot x \in I \\ &\iff p \cdot x = q \cdot x^2(y-1) \\ &\iff p = q \cdot x \cdot (y-1) \\ &\iff p \in \langle x(y-1) \rangle \end{aligned}$$

So $V(I:J) = V(x) \cup V(y-1) \supsetneq V(y-1)$.

However a similar computation gives

$$I:J^2 = \langle x^2(y-1) \rangle : \langle x^2 \rangle = \langle y-1 \rangle$$

so

$$V(I:J^2) = \overline{V(I) \setminus V(J)}$$

# IDEAL SATURATIONS

## Definition 9.45

If $I, J \subset k[x_1, \ldots, x_n]$ ideals,
then the saturation of $I$ with respect to $J$ is

$$I : J^\infty = \{ p \in k[x_1, \ldots, x_n] : \forall q \in J \ \exists N \in \mathbb{N} \ pq^N \in I \}$$

Here $J^\infty$ is merely notation, NOT an infinite product!

## Proposition 9.46

If $I, J \subset k[x_1, \ldots, x_n]$ ideals, then $I : J^\infty$ is an ideal, and

(i) $I \subset I : J \subset I : J^\infty$

(ii) $\exists N \in \mathbb{N}$ such that $I : J^\infty = I : J^N$

(iii) $\sqrt{I : J^\infty} = \sqrt{I} : J$

## Proof

(i) Since $J \supset J^2 \supset J^3 \supset \ldots$ we get

$$I : J \subset I : J^2 \subset I : J^3 \subset \ldots$$

By the Ascending Chain Condition, we find $N \in \mathbb{N}$ such that $I : J^N = I : J^{N+1}$.

If $p \in I : J^N$ then $\forall q \in J$ $q^N \in J^N$.

$\Rightarrow pq^N \in I \Rightarrow p \in I : J^\infty$.

Hence $I \subset I : J \subset I : J^N \subset I : J^\infty$.

(ii) To show the converse inclusion $I : J^\infty \subset I : J^N$,

let $J = \langle q_1, \dots, q_s \rangle$ and $p \in I : J^\infty$.

For each $i = 1, \dots, s$ $\exists N_i \in \mathbb{N}$ s.t. $p q_i^{N_i} \in I$.

Let $M = \max(N_1, \dots, N_s)$, so $p q_i^M \in I \;\; \forall i = 1, \dots, s$.

Consider the ideal $J^{sM}$:

Elements of $J$ have the form $\sum_{i=1}^{s} h_i q_i$,

so elements of $J^{sM}$ are linear combinations of

$$\prod_{j=1}^{sM} \left( \sum_{i=1}^{s} h_{ij} q_i \right) = \sum_{i_1, \dots, i_{sM} = 1}^{s} (\dots) q_{i_1} q_{i_2} \cdots q_{i_{sM}}$$

Write the products as

$$q_{i_1} q_{i_2} \cdots q_{i_{sM}} = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$$

Then $\alpha_1 + \dots + \alpha_s = sM$, so we have some $\alpha_i \geq M$.

Hence

$$J^{sM} \subset \langle q_1^M, \dots, q_s^M \rangle$$

so $p q_i^M \in I \;\; \forall i = 1, \dots, s \Rightarrow p q \in I \;\; \forall q \in J^{sM}$.

That is we have found that $p \in I : J^\infty \Rightarrow p \in I : J^{sM}$

(iii) $\underline{\sqrt{I : J^\infty} \subset \sqrt{I} : J}$

Let $p \in \sqrt{I : J^\infty}$, so $p^m \in I : J^\infty$, i.e.

$$\forall q \in J \; \exists N \in \mathbb{N} \quad p^m q^N \in I$$

Then also $(pq)^{\max(m,N)} \in I$ so $pq \in \sqrt{I}$

and we obtain $p \in \sqrt{I} : J$.

$\underline{\sqrt{I : J^\infty} \supset \sqrt{I} : J}$

Let $p \in \sqrt{I} : J$ so

$$\forall q \in J \quad pq \in \sqrt{I} \Rightarrow \forall q \in J \; \exists m \in \mathbb{N} \quad p^m q^m \in I$$

Let $m_i \in \mathbb{N}$ be the powers such that $p^{m_i} q_i^{m_i} \in I$, where $J = \langle q_1, \dots, q_s \rangle$. Let $M = \max(m_1, \dots, m_s)$

Repeating the argument in (ii), we find

$$p^M q \in I \quad \forall q \in J^{sM}$$

Hence $p^M \in I : J^{sM} \subset I : J^\infty$, so

$$p \in \sqrt{I : J^\infty}. \qquad \square$$

## Theorem 9.47

Let $I, J \subset k[x_1, \ldots, x_n]$ ideals. Then

(i) $V(I) = V(I+J) \cup V(I : J^\infty)$

(ii) $\overline{V(I) \setminus V(J)} \subset V(I : J^\infty)$

(iii) If $k$ is algebraically closed, then

$$\overline{V(I) \setminus V(J)} = V(I : J^\infty)$$

## Proof

(ii) Repeat the proof of Proposition 9.43 (ii).

$$\overline{V(I) \setminus V(J)} = V(I(V(I) \setminus V(J))) \subset V(I : J^\infty)$$

$$\Leftarrow I(V(I) \setminus V(J)) \supset I : J^\infty$$

Let $p \in I : J^\infty$ and $a \in V(I) \setminus V(J)$.

Then

$$p \in I : J^\infty \Rightarrow \forall q \in J \ \exists N \in \mathbb{N} \ pq^N \in I$$

$$a \in V(I) \Rightarrow \forall q \in J \ \exists N \in \mathbb{N} \ p(a) q^N(a) = 0$$

$$a \notin V(J) \Rightarrow \exists q \in J \ q(a) \neq 0$$

$$\Rightarrow p(a) = 0 \Rightarrow p \in I(V(I) \setminus V(J)).$$

(i) $V(I) = (V(I) \cap V(J)) \cup (V(I) \setminus V(J))$

$\subset V(I+J) \cup (\overline{V(I) \setminus V(J)})$

$\subset V(I+J) \cup V(I : J^\infty)$ by (ii)

$\subset V(I+J) \cup V(I : J)$ by Prop 9.46

$= V(I)$ by Prop 9.43

(iii) When $k$ is algebraically closed, the ideal-variety correspondence gives

$$V(I:J^\infty) \subset \overline{V(I)\setminus V(J)} = V(I(V(I)\setminus V(J)))$$

$$\iff \sqrt{I:J^\infty} \supset \sqrt{I(V(I)\setminus V(J))}$$

$$\iff \sqrt{I}:J \supset I(V(I)\setminus V(J))$$

where the latter equivalence follows from Proposition 9.4.6(iii) and the fact that $I(S)$ is a radical ideal for any set $S \subset k^n$.

Let $p \in I(V(I)\setminus V(J))$. If $q \in J$, then $\forall a \in V(I)$

either $a \in V(J) \Rightarrow q(a) = 0$

or $a \in V(I)\setminus V(J) \Rightarrow p(a) = 0$

So in any case $p(a)q(a) = 0$, and we get

$pq \in I(V(I))$

Nullstellensatz $\Rightarrow pq \in \sqrt{I} \Rightarrow p \in \sqrt{I}:J$. $\square$