Theorem 9.61

Let k be an algebraically closed field.

Let $I \subset k[x_1, \ldots, x_n]$ be a maximal ideal.

Then $\exists a_1, \ldots, a_n \in k$ such that $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Proof

By the Weak Nullstellensatz (Theorem 9.2)

$$I \neq k[x_1, \ldots, x_n] \Rightarrow V(I) \neq \emptyset$$

Let $a = (a_1, \ldots, a_n) \in V(I) \subset k^n$. Then we have

$$I \subset I(V(I)) \subset I(\{a\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle.$$

By maximality $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. $\square$

Corollary 9.62

Let k be an algebraically closed field.

Then we have the bijective correspondence

$$\{\text{points}\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{maximal ideals}\}$$

Proof

By Prop 9.60 and Thm 9.61, maximal ideals are exactly the ideals $\langle x_1 - a_1, \ldots, x_n - a_n \rangle = I(\{(a_1, \ldots, a_n)\})$ $\square$

Recall from Exercise 5.1 the descending chain condition for varieties

$$V_1 \supset V_2 \supset V_3 \supset \dots \quad \Rightarrow \exists N \in \mathbb{N} \quad V_N = V_{N+1} = \dots$$

Theorem 9.63

Let $V \subset k^n$ be a variety.

Then $\exists V_1, \dots, V_m$ irreducible varieties such that

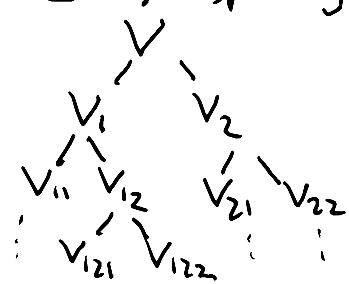
$$V = V_1 \cup \dots \cup V_m$$

Proof

If V is not irreducible, then $V = V_1 \cup V_2$ for some smaller varieties $V_1 \subsetneq V$ and $V_2 \subsetneq V$.

If V_1 is not irreducible then we can further decompose $V_1 = V_{11} \cup V_{12}$ with $V_{11} \subsetneq V$ and V_{12}.

Form a tree by splitting reducible varieties as above.



By the descending chain condition the tree cannot be infinite, since that would give an infinite chain

$$V \supsetneq V_a \supsetneq V_{ab} \supsetneq V_{abc} \supsetneq \dots$$

The leaves of the tree give the finite union of irreducibles $\square$

Example 9.64

Consider $V = V((x^3-x)(xz-y^2),\ (x^3-x)(x^3-yz)) \subset \mathbb{R}^3$

Splitting up the $x^3 - x = x(x^2-1) = x(x-1)(x+1)$ factor we obtain

$$\begin{tikzcd}[column sep=small]
 & & & V \arrow[dll] \arrow[drr] & & \\
 & V(x^3-x) \arrow[dl] \arrow[dr] & & & & V(xz-y^2,\ x^3-yz) =: W \\
V(x) & & V(x^2-1) \arrow[dl] \arrow[dr] & & & \\
 & V(x-1) & & V(x+1) & &
\end{tikzcd}$$

However W is not irreducible:

We observe that $V(x,y) \subsetneq W$.

To find a decomposition $\underline{W = V(x,y) \cup W_2}$,

consider $W_2 = \overline{W \setminus V(x,y)}$.

By Theorem 9.47 $W_2 = V(I : J^\infty)$, where

$$I = \langle xz - y^2,\ x^3 - yz \rangle, \quad J = \langle x, y \rangle$$

Moreover I is radical (we will prove this later),

so Proposition 9.46 $\Rightarrow$ $I : J^\infty = I : J$.

Computing with the method described in Algorithm 9.50, we find

$$I : \langle x \rangle = I + \langle x^2 y - z^2 \rangle = I : \langle y \rangle$$

so $I : J = (I : \langle x \rangle) \cap (I : \langle y \rangle) = \langle xz - y^2,\ x^3 - yz,\ x^2 y - z^2 \rangle$

and $W = V(x,y) \cup V(xz-y^2,\ x^3-yz,\ x^2y - z^2)$

To show $V(I:J)$ irreducible, we claim that

$$V(I:J) = \{(t^3, t^4, t^5) : t \in \mathbb{R}^3\}$$

so $V(I:J)$ irreducible by Proposition 9.57.

$(t^3, t^4, t^5) \in V(I:J)$ is quick to verify.

For the converse let $(x,y,z) \in V(I:J)$ and set $t = \sqrt[3]{x}$.

Solving y, z from

$$t^3 z - y^2 = t^9 - yz = t^6 y - z^2 = 0$$

we find $y = t^4$ and $z = t^5$.

(In fact, for the lex order $z > y > t$ $y^3 - t^{12}$ is in the Gröbner basis and for lex order $y > z > t$, $z^3 - t^{15}$ is in the basis)

Definition 9.65

Let V be a variety. A decomposition

$$V = V_1 \cup \dots \cup V_m \text{ into irreducibles is}$$

a <u>minimal decomposition</u> if $V_i \not\subset V_j$ for all $i \neq j$.

(also called an <u>irredundant union</u>)

The V_i are called the irreducible components of V.

Theorem 9.66

A minimal decomposition of a variety V is unique up to reordering terms.

Proof

From Theorem 9.63 we obtain some decomposition $V = V_1 \cup \dots \cup V_n$ into irreducibles. Removing those V_i for which $V_i \subset V_j$ for some $i \neq j$ gives a minimal decomposition.

Suppose $V = V_1' \cup \dots \cup V_l'$ is another minimal decomposition.

Since $V_i \subset V_i \cap V = V_i \cap V_1' \cup \dots \cup V_i \cap V_l'$

and V_i is irreducible $\exists j$ with $V_i = V_i \cap V_j'$

Similarly for $V_j' \subset V_j' \cap V = V_j' \cap V_1 \cup \dots \cup V_j' \cap V_m$

$\exists k$ with $V_j' = V_k \cap V_j'$. Then

$$V_i \subset V_j' \subset V_k$$

so minimality implies $V_i = V_j'$.

This argument gives an injection

$$\{V_1, \dots, V_m\} \longrightarrow \{V_1', \dots, V_l'\}$$

Repeating with the roles V_i' and V_j reversed shows the decompositions are the same up to reordering $\square$

<u>Proposition 9.67</u>

Let $W \subsetneq V$ be varieties,

Then $V \setminus W$ is Zariski dense in V if and only if W does not contain any irreducible component of V.

<u>Proof</u>

"$\Leftarrow$" Let $V = V_1 \cup \dots \cup V_m$ be the minimal decomposition of V.

By assumption $\forall i$ $V_i \not\subset W$, so $V_i \cap W \subsetneq V_i$.

Then $V_i = (V_i \cap W) \cup \overline{V_i \setminus W} \Rightarrow \overline{V_i \setminus W} = V_i$

Hence $\overline{V \setminus W} \underset{\text{Lemma 9.38 (iii)}}{=} \overline{V_1 \setminus W} \cup \dots \cup \overline{V_m \setminus W} = V_1 \cup \dots \cup V_m = V$.

"$\Rightarrow$" If $V_i \subset W$, then

$$\overline{V \setminus W} = \overline{V_1 \setminus W} \cup \dots \cup \underbrace{\overline{V_i \setminus W}}_{=\emptyset} \cup \dots \cup \overline{V_m \setminus W}$$

$$\subset V_1 \cup \dots \cup V_{i-1} \cup V_{i+1} \cup \dots \cup V_m \neq V,$$

where "$\neq V$" follows since $V_1 \cup \dots \cup V_{i-1} \cup V_{i+1} \cup \dots \cup V_m$ is a decomposition into irreducibles with $V_j \subsetneq V_k$ if $j \neq k$, different from $V_1 \cup \dots \cup V_m$, but the minimal decomposition of V is unique. $\square$

<u>Definition 9.68</u>

Let $I \subset k[x_1, \dots, x_n]$ be a radical ideal. A decomposition $I = P_1 \cap \dots \cap P_m$ into prime ideals P_i is a <u>minimal decomposition</u> if $P_i \not\subset P_j$ for $i \neq j$. (also called an <u>irredundant intersection</u>)

<u>Theorem 9.69</u>

If k is algebraically closed, the minimal decomposition of a radical ideal is unique up to reordering.

<u>Proof</u>

Follows from the unique decomposition of varieties (9.63 and 9.66) by the ideal-variety correspondence:

Thm 9.9: $V(I) \leftrightarrow I$, Thm 9.34: $V(I \cap J) = V(I) \cup V(J)$,

and Prop 9.55: irreducible variety $\leftrightarrow$ prime ideal

Theorem 9.70

If K is algebraically closed and $I = P_1 \cap \dots \cap P_r \subsetneq k[x_1, \dots, x_n]$ is the minimal decomposition of a radical ideal I, then $\exists q_1, \dots, q_r \in k[x_1, \dots, x_n]$ such that $P_i = I : q_i$ and conversely if $I:q$ is a proper prime ideal, then $I:q = P_i$ for some $i \in \{1, \dots, r\}$.

Proof

First we observe that for a prime ideal P

$$q \in P \Rightarrow P:q = \langle 1 \rangle \quad (\text{since } pq \in P \ \forall p) \text{ and}$$

$$q \notin P \Rightarrow P:q = P \quad (\text{since } pq \in P \Rightarrow p \in P)$$

Therefore

$$I:q = (P_1 \cap \dots \cap P_r):q \overset{(pq \in P_1 \cap \dots \cap P_r \iff pq \in P_i \ \forall i)}{=} P_1:q \cap \dots \cap P_r:q = \bigcap_{q \notin P_i} P_i$$

Let $q_i \in \bigcap_{i \neq j} P_j \setminus P_i$ (which is nonempty by minimality)

Then $I:q_i = P_i$ by the above.

For the converse statement, it suffices to show that an intersection $J \cap H$ of ideals with $J \not\subset H$, $H \not\subset J$ is not prime:

Take $p \in J \setminus H$ and $q \in H \setminus J$.

Then $pq \in J \cap H$ but $p \notin J \cap H$ and $q \notin J \cap H$. $\square$