10. POLYNOMIAL MAPPINGS ON A VARIETY

Definition 10.1 Let $V \subset K^m$, $W \subset K^n$ be varieties.

- A mapping $\phi : V \to W$ is a *polynomial mapping* if $\exists\, p_1, \dots, p_n \in K[x_1, \dots, x_m]$ such that
  $\phi(a) = (p_1(a), \dots, p_n(a)) \quad \forall a \in V$
- The tuple $(p_1, \dots, p_n)$ *represents* ϕ
- The polynomials p_i are components of this representation

Example 10.2

(1) polynomial parametrization:

$$p_1 = t, \quad p_2 = t^2, \quad p_3 = t^3, \quad \phi = (p_1, p_2, p_3)$$

gives a polynomial mapping

$$\phi : K \to V(y - x^2, z - x^3).$$

(2) projection:

$$p_1 = y, \quad p_2 = z \quad \text{in} \quad K[x, y, z], \quad \phi = (p_1, p_2)$$

on $V(y - x^2, z - x^3)$ gives a polynomial mapping

$$\phi : V(y - x^2, z - x^3) \to V(y^3 - z^2)$$

(3) $q_1 = y, \quad q_2 = xy \quad$ in $K[x, y, z]$, $\psi = (q_1, q_2)$

also gives a polynomial mapping

$$\psi : V(y - x^2, z - x^3) \longrightarrow V(y^3 - z^2)$$

In fact, if $a = (t, t^2, t^3) \in V(y - x^2, z - x^3)$ then

$$\psi(a) = (q_1(a), q_2(a)) = (t^2, t \cdot t^2) = (t^2, t^3) = \phi(a)$$

Proposition 10.3

Let $V \subset k^m$ be a variety.

(i) $p, q \in k[x_1, \dots, x_m]$ represent the same polynomial mapping $\phi : V \to k$ if and only if $p - q \in I(V)$

(ii) $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ represent the same polynomial mapping $\phi : V \to k^n$ if and only if $p_i - q_i \in I(V)$, $1 \le i \le n$.

Proof

(i) $p(a) = q(a) \quad \forall a \in V \iff (p-q)(a) = 0 \quad \forall a \in V$

$$\iff p - q \in I(V)$$

(ii) Apply (i) componentwise $\square$

Definition 10.4

- The <u>coordinate ring</u> of a variety $V \subset k^n$ is the quotient ring
$$k[V] = k[x_1, \dots, x_n] / I(V)$$
- The equivalence class of $p \in k[x_1, \dots, x_n]$ in $k[V]$ will be denoted by $[p]$.

Prop 10.3 implies the bijective correspondence

$$k[V] \longleftrightarrow \{\text{polynomial mappings } V \to k\}$$

Proposition 10.5

Let $V \subset k^n$ be a variety. Then

$$V \text{ irreducible} \iff k[V] \text{ is an integral domain}$$
$$(\phi \cdot \psi = 0 \Rightarrow \phi = 0 \text{ or } \psi = 0)$$

Proof

"$\Rightarrow$" Let $\phi : V \to k$ and $\psi : V \to k$ polynomial mappings, $\phi \cdot \psi = 0$.
Take representatives $p, q \in k[x_1, \dots, x_n]$ of ϕ, ψ, so
$p(a)q(a) = 0 \ \forall a \in V \Rightarrow V = \big(V \cap V(p)\big) \cup \big(V \cap V(q)\big)$.

V irreducible $\Rightarrow$ $V \subset V(p) \Rightarrow p \in I(V) \Rightarrow \phi = 0$
or
$V \subset V(q) \Rightarrow q \in I(V) \Rightarrow \psi = 0$

"$\Leftarrow$" Suppose V not irreducible, so $V = V_1 \cup V_2$, $V_1 \subsetneq V$, $V_2 \subsetneq V$.
Then $\exists p \in I(V_1) \setminus I(V)$ and $q \in I(V_2) \setminus I(V)$, so
$p(a)q(a) = 0 \ \forall a \in I(V)$
$\Rightarrow \phi \cdot \psi = 0$ for the polynomial mappings $\phi, \psi : V \to k$
with representatives p, q,
but $\phi \neq 0$ on and $\psi \neq 0$. $\square$

Example 10.6

Let $V = V(p_1, p_2, p_3) \subset \mathbb{C}^3$, where

$$p_1 = x^2 + 2xz + 2y^2 + 3y$$
$$p_2 = xy + 2x + z$$
$$p_3 = xz + y^2 + 2y$$

Claim: $\mathbb{C}[V]$ has a bijection with $\mathbb{C}[x]$.

Proof: A Gröbner basis of $\langle p_1, p_2, p_3 \rangle$ in lex order $y > z > x$

$$g_1 = y - x^2$$
$$g_2 = z + x^3 + 2x$$

Consider the polynomial mappings

$$\pi : V \longrightarrow \mathbb{C} \qquad \pi(x, y, z) = x$$
$$\phi : \mathbb{C} \rightarrow V \qquad \phi(x) = (x, x^2, -x^3 - 2x)$$

Then π and ϕ are inverse mappings:

$$\pi \circ \phi : \mathbb{C} \rightarrow \mathbb{C}, \quad \pi \circ \phi(x) = x$$
$$\phi \circ \pi : V \rightarrow V, \quad \phi \circ \pi(x, y, z) = (x, x^2, -x^3 - 2x)$$
$$= (x, x^2 + g_1, -x^3 - 2x + g_2)$$
$$= (x, y, z) \text{ on } V$$

This implies that

$$\mathbb{C}[V] \rightarrow \mathbb{C}[x], \quad [p] \longmapsto p \circ \phi$$

is a well defined map and

$$\mathbb{C}[x] \rightarrow \mathbb{C}[V], \quad p \mapsto [p \circ \pi]$$

is its inverse map.

<u>Definition 10.7</u> Let V, W be varieties.
Let $\alpha : V \to W$ be a polynomial mapping.
The <u>pullback mapping</u> of α is
$$\alpha^* : K[W] \to K[V], \quad \alpha^*(\phi) = \phi \circ \alpha$$

<u>Proposition 10.8</u>
(i) The pullback mapping $\alpha^* : K[W] \to K[V]$ is a ring homomorphism and $\alpha^*([c]) = [c]$ any constant polynomial c (representing a constant mapping)
(ii) Let $\Phi : K[W] \to K[V]$ be a ring homomorphism such that $\Phi([c]) = [c]$ for any constant polynomial c.
Then there is a unique polynomial mapping $\alpha : V \to W$ such that $\Phi = \alpha^*$.

<u>Proof</u>
(i) If $\phi : W \to K$ polynomial mapping,
then $\alpha^*(\phi) = \phi \circ \alpha : V \to K$ polynomial mapping.
That α^* is a ring homomorphism follows by
$$(\phi_1 + \phi_2) \circ \alpha = \phi_1 \circ \alpha + \phi_2 \circ \alpha$$
$$(\phi_1 \phi_2) \circ \alpha = (\phi_1 \circ \alpha)(\phi_2 \circ \alpha)$$
If $c \in K$ is a constant, the constant mapping
$$\phi : W \to K, \quad \phi(a) = c \quad \forall a \in W$$
has the pullback $\alpha^* \phi : V \to K$, $\alpha^* \phi(a) = \phi(\alpha(a)) = c \quad \forall a \in V$.

(ii) Let $V \subset k^m$ and $W \subset k^n$, and consider

$$K[V] = K[x_1, \ldots, x_m] / I(V)$$
$$k[W] = K[y_1, \ldots, y_n] / I(W)$$

Consider the Φ-images

$$[y_i] \in K[W] \longmapsto \Phi([y_i]) \in K[V]$$

and take representatives $p_i \in k[x_1, \ldots, x_n]$ of $\Phi([y_i])$.

Define the polynomial mapping $\alpha = (p_1, \ldots, p_n)$

<u>Claim</u> α is a map $V \to W$ and $\alpha^* = \Phi$.

Proof: First we show $[q \circ \alpha] = \Phi([q]) \in K[V]$ $\forall [q] \in k[W]$.

$(q \mapsto q \circ \alpha)$ and Φ are both ring homomorphisms,

so it suffice to check that $[q \circ \alpha] = \Phi([q])$

for generators $[q]$ of $K[W]$ as a ring.

The ring $K[y_1, \ldots, y_n]$ is generated by the constants and

the monomials $y_1, \ldots, y_n$ since every polynomial is a sum

$$\sum c_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \quad c_\alpha \in K, \ (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$$

Hence $K[W]$ is generated as a ring by

the constant mappings and $[y_1], \ldots, [y_n] \in K[W]$.

For constants $c \in K$

$$[c \circ \alpha] = [c] = \Phi([c])$$

For the coordinate generators $y_i \in K[y_1 - y_n]$

$$[y_i \circ \alpha] = [p_i] = \Phi([y_i])$$

Proving $[q \circ \alpha] = \Phi([q])$.

If $q \in I(W)$, then $[q] = 0 \in K[W]$.
Then $0 = \Phi([q]) = [q \circ \alpha] \in K[V]$, so

$$q \circ \alpha \in I(V) \quad \forall q \in I(W).$$

$\Rightarrow$ if $a \in V$ then $\alpha(a) \in V(I(W)) = W \Rightarrow \alpha(V) \subset W$.

That is, α is a mapping $V \to W$.

Finally, by definition of the pullback

$$\alpha^*([q]) = [q \circ \alpha] = \Phi([q]) \quad \forall [q] \in K[W],$$

so $\alpha^* = \Phi$.

To show uniqueness of $\alpha : V \to W$, let $\beta : V \to W$ be another polynomial mapping represented by $(q_1, \ldots, q_n)$ with $\beta^* = \Phi$. Then

$$[q_i] = \beta^*([y_i]) = \Phi([y_i]) = \alpha^*([y_i]) = [p_i] \in K[V]$$

$\Rightarrow [p_i - q_i] = 0 \in K[V] \Rightarrow p_i - q_i \in I(V)$

Prop 10.3 $\Rightarrow$ α and β define the same mapping $V \to W$ $\square$

Definition 10.9

Varieties $V \subset k^m$ and $W \subset k^n$ are isomorphic

if $\exists \alpha : V \to W$, $\beta : W \to V$ polynomial mappings such that

$\alpha \circ \beta = id_W$ and $\beta \circ \alpha = id_V$

the identity mapping $(x_1 \ldots x_n) \mapsto (x_1 \ldots x_n)$ on W

Theorem 10.10

Varieties V and W are isomorphic if and only if

$\exists\, \Phi : k[V] \to k[W]$ ring isomorphism with $\Phi([c]) = [c]$

the identity map on constants. $c \in k$.

Proof

"$\Rightarrow$" Let $\alpha : V \to W$ and $\beta : W \to V$ inverse polynomial mappings.

Then $\alpha \circ \beta = id_W$, so for all $\phi \in k[W]$

$$(\beta^* \circ \alpha^*)(\phi) = \beta^*(\alpha^* \phi) = \beta^*(\phi \circ \alpha) = \underbrace{\phi \circ \alpha \circ \beta}_{= (\alpha \circ \beta)^* \phi} = \phi$$

Similarly $(\alpha^* \circ \beta^*)(\phi) = \phi \quad \forall \phi \in k[V]$,

so $\alpha^* : k[W] \to k[V]$ and $\beta^* : k[V] \to k[W]$ are

inverse ring homomorphisms. Take $\Phi = \beta^*$.

"$\Leftarrow$" Prop 10.8 $\Rightarrow$ $\Phi = \beta^*$ and $\Phi^{-1} = \alpha^*$ for some

polynomial mappings $\alpha : V \to W$ and $\beta : W \to V$.

Then $(\alpha \circ \beta)^* = \beta^* \circ \alpha^* = \Phi \circ \Phi^{-1} = id_{k[W]} = (id_W)^*$

By the uniqueness in Prop 10.8 $\alpha \circ \beta = id_W$,

and similarly $\beta \circ \alpha = id_V$ $\square$