Definition 10.17

Let $I \subset k[x_1, \ldots, x_n]$ be an ideal.
If I satisfies any of the equivalent conditions (i)–(iv) of the Finiteness Theorem 10.16 (e.g. $k[x_1, \ldots, x_n]/I$ finite dim) then I is <u>zero dimensional</u>.

Proposition 10.18

Let k be algebraically closed and $I \subset k[x_1, \ldots, x_n]$ radical ideal.
Then
$$\underset{\substack{\uparrow \\ \text{number of points}}}{\#V(I)} = \dim_k \left(k[x_1, \ldots, x_n]/I \right)$$

Proof

Theorem 10.16 implies $V(I)$ finite $\iff$ $k[x_1, \ldots, x_n]/I$ finite dim.
Here it suffices to consider zero dimensional ideals I, so
$$V(I) = \{a_1, \ldots, a_m\} \subset k^n \text{ by Theorem 10.16.}$$
First we choose polynomials $p_1, \ldots, p_m \in k[x_1, \ldots, x_n]$ s.t.
$$p_i(a_i) \neq 0, \quad p_i(a_j) = 0 \quad \forall j \neq i$$

(These exist since finite sets are Zariski closed.
Recall: Zariski closure of $\{a_1, \ldots, a_{n-1}\}$
is the set of points $b \in k^n$ such that
$\forall p \in k[x_1, \ldots, x_n]$: $p(a_1) = \ldots = p(a_{n-1}) = 0 \Rightarrow p(b) = 0$)

Claim: $[p_1], \ldots, [p_m]$ is a K-basis of $K[x_1, \ldots, x_n]/I$.

K-linear independence: Suppose

$$0 = \sum_i c_i [p_i] = \left[\sum_i c_i p_i\right], \quad c_1, \ldots, c_m \in K$$

Then $q = \sum_i c_i p_i \in I$, so it vanishes at all $a_j \in V(I)$.

$\Rightarrow 0 = q(a_j) = \sum_i c_i p_i(a_j) = c_j p_j(a_j)$

$\Rightarrow c_j = 0$ since $p_j(a_j) \neq 0$

$[p_1], \ldots, [p_m]$ span everything:

Let $[q] \in K[x_1, \ldots, x_n]/I$ and define $c_i = \dfrac{q(a_i)}{p_i(a_i)} \in K$.

Then $q(a_j) = c_j p_j(a_j) = \sum_i c_i p_i(a_j) \quad \forall j$,

so $q - \sum_i c_i p_i \in I(\{a_1, \ldots, a_n\}) = I(V(I)) \underset{\text{Nullstellensatz}}{=} \sqrt{I} \overset{I \text{ radical}}{=} I$

That is,

$$[q] = \sum_i c_i [p_i]$$

so $[p_1], \ldots, [p_m]$ span.

Hence $\dim_K \left(K[x_1, \ldots, x_n]/I\right) = m = \# V(I)$ $\square$

Example 10.19

Let I be a zero dimensional ideal, e.g.

$$I = \langle 2y+z-1,\ z^2-1,\ xz-x-z+1,\ x^2+x+z-1 \rangle \subset \mathbb{Q}[x,y,z]$$

I is zero dimensional since $x^2, y, z^2 \in \langle LT(J) \rangle$ in lex.

To find the points $V(I)$, apply the Elimination & Extension Theorems, which take a simpler form for 0-dim ideals.

A reduced Gröbner basis G in lex is

$$g_1 = x^2 + x + z - 1$$

$$g_2 = xz - x - z + 1$$

$$g_3 = y + \tfrac{1}{2}z - \tfrac{1}{2}$$

$$g_4 = z^2 - 1$$

I zero dimensional & G reduced

$\Rightarrow$ $G \cap \mathbb{Q}[z]$ is a singleton, $\{g_4\}$ in this case.

We find the roots $a_3 = \pm 1 \in \mathbb{Q}$.

For each root $a_3 \in \mathbb{Q}$, we substitute

$$G(a_3) = \{ g(x,y,a_3) : g \in G \}$$

I zero dimensional $\Rightarrow$ $LM(g) = y^m$ for some $g \in G$.

Thm 8.8 (used in the proof of the extension theorem)

$\Rightarrow$ $I(z=a_3)$ generated by $g(y,a_3)$ with minimal y-degree.

In this case for both $a_3 = \pm 1$ we have $g = g_3$, giving

$$g_3(y, +1) = y \qquad \text{or} \qquad g_3(y,-1) = y-1$$

We get partial solutions $(a_2, a_3) = (0,1)$ and $(1,-1)$

For each partial solution, we again substitute

$$G(a_2, a_3) = \{ g(x, a_2, a_3) : g \in G \}$$

and zero dimensionality implies extension determined by a single Gröbner basis element of minimal x-degree.

$$\boxed{g_1(x,0,1) = x^2 + x} \qquad \text{or} \qquad g_1(x,1,-1) = x^2 + x - 2$$

$$g_2(x,0,1) = 0 \qquad \qquad \boxed{g_2(x,1,-1) = -2x+2}$$

So we obtain full solutions from the x-roots:

$$x^2 + x = x(x+1) = 0 \qquad \text{or} \qquad -2x+2 = -2(x-1) = 0$$

Thus $V(I) = \{(0,0,1), (-1,0,1), (1,1,-1)\}$, which we reconstructed as

$$(*, *, *)$$

$$z^2 - 1$$

$$(*, *, 1) \qquad \qquad (*, *, -1)$$

$$y = 0 \qquad \qquad y - 1 = 0$$

$$(*, 0, 1) \qquad \qquad (*, 1, -1)$$

$$x^2 + x = 0 \qquad \qquad -2x + 2 = 0$$

$$(0,0,1) \quad (-1,0,1) \qquad \qquad (1,1,-1).$$

Lemma 10.20

Let $V \subset k^n$ be a variety.
Then there is a bijective correspondence

$$\{\text{ideals } I \subset k[V]\} \longleftrightarrow \left\{\begin{array}{l}\text{ideals } J \subset k[x_1, \dots, x_n] \text{ with} \\ I(V) \subset J \subset k[x_1, \dots, x_n]\end{array}\right\}$$

Proof

If $I \subset k[V]$ ideal, define $J \subset k[x_1, \dots, x_n]$ by

$$J = \{ p \in k[x_1, \dots, x_n] : [p] \in I \}$$

Then J is an ideal: $\forall p, q \in J$ and $h \in k[x_1, \dots, x_n]$

$$0 \in J, \quad [p] + [q] \in I \Rightarrow [p+q] \in I, \quad [p][h] \in I \Rightarrow [ph] \in I$$

Moreover $I(V) \subset J$, since $[p] = 0 \in I \quad \forall p \in I(V)$.

Conversely, given an ideal $J \supset I(V)$, define $I \subset k[V]$ by

$$I = \{ [p] \in k[V] : p \in J \}.$$

Then I is an ideal: $\forall [p], [q] \in I$ and $[h] \in k[V]$

$$0 \in J \Rightarrow [0] \in I, \quad pq \in J \Rightarrow [p][q] \in I, \quad p+q \in J \Rightarrow [p]+[q] \in I$$

We obtain a correspondence $I \subset k[V] \Leftrightarrow I(V) \subset J \subset k[x_1, \dots, x_n]$

$$[p] \in I \iff p \in J \qquad \square$$

Definition 10.21

Let $W \subset K^n$ be a variety.

- For an ideal $J \subset K[V]$, define
$$V_W(J) = \{ a \in V : \phi(a) = 0 \ \forall \phi \in J \}$$
This is called a subvariety of W.

- For a subset $U \subset W$, define
$$I_W(U) = \{ \phi \in K[V] : \phi(a) = 0 \ \forall a \in U \}$$

Proposition 10.22

Let $W \subset K^n$ be a variety.

(i) $J \subset K[V]$ ideal $\Rightarrow$ $V_W(J)$ is a variety contained in W.

(ii) $U \subset W$ subset $\Rightarrow$ $I_W(U) \subset K[V]$ ideal

(iii) $J \subset K[V]$ ideal $\Rightarrow$ $J \subset \sqrt{J} \subset I_W(V_W(J))$

(iv) $U \subset W$ subvariety $\Rightarrow$ $U = V_W(I_W(U))$

Proof

(i) Lemma 10.20 $\Rightarrow$ $\tilde{J} = \{ p \in K[x_1, \ldots, x_n] : [p] \in J \} \supset I(W)$.

Then $V(\tilde{J}) \subset W = V(I(W))$ and
$$V(\tilde{J}) = \{ a \in V : p(a) = 0 \ \forall p \in \tilde{J} \}$$
$$= \{ a \in V : [p](a) = 0 \ \forall p \in J \} = V_W(J)$$

The proofs of (ii)–(iv) identical to the proofs of the corresponding statements (ii) $\leftrightarrow$ Lemma 4.9, (iii) $\leftrightarrow$ Lemma 9.6 + Theorem 9.3, (iv) $\leftrightarrow$ Theorem 9.9(ii) $\square$

Lemma 10.23

$J \subset k[V]$ radical $\iff$ $\tilde{J} = \{p \in k[x_1, \ldots, x_n] : [p] \in J\}$ radical

Proof

$p^m \in \tilde{J} \iff [p]^m \in J$ $\square$

Theorem 10.24

Let k be algebraically closed and $W \subset k^n$ variety.

(i) Nullstellensatz in $k[V]$: If $J \subset k[V]$ ideal, then

$$I_W(V_W(J)) = \sqrt{J}$$

(ii) The maps

$$\{\text{subvarieties } U \subset W\} \underset{V_W}{\overset{I_W}{\rightleftarrows}} \{\text{radical ideals } J \subset k[V]\}$$

are inclusion-reversing inverse bijections.

(iii) $$\{\text{points } a \in W\} \underset{V_W}{\overset{I_W}{\rightleftarrows}} \{\text{maximal ideals } J \subset k[V]\}$$

are also bijections.

Proof

(i) Using the correspondence $J \subset k[V] \overset{\text{Lemma 10.20}}{\longleftrightarrow} I(V) \subset \tilde{J} \subset k[x_1, \ldots, x_n]$,

$$I_W(V_W(J)) \underset{\text{as in Prop 10.22}}{\longleftrightarrow} I(V(\tilde{J})) \underset{\text{Nullstellensatz}}{=} \sqrt{\tilde{J}} \underset{\text{Lemma 10.23}}{\longleftrightarrow} \sqrt{J}$$

(ii) Follows from (i) and Prop 10.22(iv).

(iii) Using Cor 9.62:

$$\begin{array}{ccc} J \text{ maximal} & & J = \langle [x_1]-a_1, \ldots, [x_n]-a_n \rangle \\ \updownarrow & & \updownarrow \\ \tilde{J} \text{ maximal} & \iff & \tilde{J} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle, \ (a_1, \ldots, a_n) \in V \end{array}$$ $\square$

Definition 10.25

Let $V \subset K^n$ be an irreducible variety.
The function field of V (or field of rational functions of V) is
$$K(V) = \left\{ \frac{\phi}{\psi} : \phi, \psi \in K[V], \psi \neq 0 \right\}$$
$$= \left\{ \frac{[p]}{[q]} : p, q \in K[x_1, \dots, x_n], q \notin I(V) \right\}$$

formal fractions

where $K(V)$ is equipped with the addition
$$\frac{\alpha}{\beta} + \frac{\gamma}{\delta} = \frac{\alpha\delta + \beta\gamma}{\beta\delta}$$
and multiplication
$$\frac{\alpha}{\beta} \cdot \frac{\gamma}{\delta} = \frac{\alpha\gamma}{\beta\delta}$$
and two formal fractions represent the same element when
$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \iff \alpha\delta = \beta\gamma \text{ in } K[V]$$

Note: irreducibility of V is necessary for addition and multiplication to be well defined since otherwise $\beta \neq 0 \neq \delta \not\Rightarrow \beta\delta \neq 0$.
(see Prop 10.5)

Example 10.26

Example 10.12 $\Rightarrow$ $V(y^5-x^2) \subset \mathbb{R}^2$ not isomorphic to $\mathbb{R}$

Since there is no surjective ring homomorphism $\mathbb{R}[V] \to \mathbb{R}[t]$.

y^5-x^2 irreducible $\Rightarrow$ V is irreducible.

Claim: $\exists$ field isomorphism $\mathbb{R}(V) \to \mathbb{R}(t)$

Consider the polynomial mapping

$$\beta: \mathbb{R} \to V, \qquad \beta(t) = (t^5, t^2)$$

and the rational function

$$\alpha: V \setminus \{(0,0)\} \to \mathbb{R}, \quad \alpha(x,y) = \frac{x}{y^2}$$

They give inverse maps $V \setminus \{(0,0)\} \longleftrightarrow \mathbb{R} \setminus \{0\}$

Define the pullbacks

$$\alpha^*: \mathbb{R}(t) \to \mathbb{R}(V), \quad \alpha^*\phi(x,y) = \phi\left(\frac{x}{y^2}\right)$$

$$\beta^*: \mathbb{R}(V) \to \mathbb{R}(t), \quad \beta^*\psi(t) = \psi(t^5, t^2)$$

(Even though α not defined everywhere, α^* is well defined)

We compute for $\phi \in \mathbb{R}(t)$ and $\psi \in \mathbb{R}(V)$

$$(\alpha^* \circ \beta^* \psi)(x,y) = (\beta^*\psi)\left(\frac{x}{y^2}\right) = \psi\left(\frac{x^5}{y^{10}}, \frac{x^2}{y^4}\right)$$

$$\xrightarrow{x^2 = y^5 \text{ on } V} = \psi\left(\frac{x^5}{x^4}, \frac{y^5}{y^4}\right) = \psi(x,y)$$

$$(\beta^* \circ \alpha^* \phi)(t) = (\alpha^*\phi)(t^5, t^2) = \phi\left(\frac{t^5}{t^4}\right) = \phi(t),$$

so α^* and β^* are inverse field homomorphisms.