Definition 10.27

Let $V \subset k^m$, $W \subset k^n$ be irreducible varieties.
A rational mapping $\phi : V \dashrightarrow W$ (dashed arrow)
is a mapping represented by

$$\phi(x) = (q_1(x), \dots, q_n(x)), \quad q_i = \frac{f_i}{g_i} \in k(x_1, \dots, x_m)$$

such that

(i) ϕ is defined at some point of V

(ii) if ϕ defined at $a \in V$, then $\phi(a) \in W$.

Remark • condition (i) implies ϕ defined on a Zariski-dense subset. Indeed $(q_1, \dots, q_n)$ with $q_i = f_i/g_i$ is defined on

$$k^m \setminus U \qquad U = V(g_1 \cdots g_n).$$

(i) $\Rightarrow$ $\exists a \in V \setminus U$, so $V \cap U \subsetneq V$. Then

$$V = (V \cap U) \cup \overline{(V \setminus U)} \underset{\substack{\uparrow \\ V = \overline{V \setminus U} \text{ by irreducibility}}}{\Longrightarrow} V \setminus U \text{ Zariski dense}$$

• different representatives can have different domains of definition, e.g. $\frac{f}{g} = \frac{x}{1}$ defined everywhere in $\mathbb{R}$ but $\frac{f}{g} = \frac{x^2}{x}$ defined on $\mathbb{R} \setminus \{0\}$

Definition 10.28

Two rational mappings $\phi, \psi : V \dashrightarrow W$ are equal, denoted $\phi = \psi$ as usual, if they have representatives $\phi = (q_1, \ldots, q_n)$ defined on $\emptyset \neq V \setminus U_1 \subset V$ and $\psi = (r_1, \ldots, r_n)$ defined on $\emptyset \neq V \setminus U_2 \subset V$ such that $q_i = r_i$ on $(V \setminus U_1) \cap (V \setminus U_2) = V \setminus (U_1 \cup U_2)$, $i = 1, \ldots, n$.

Proposition 10.29

Let $\phi = \left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n}\right) : V \dashrightarrow W$ and $\psi = \left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right) : V \dashrightarrow W$ rational mappings.

Then $\phi = \psi$ if and only if $f_i q_i - p_i g_i \in I(V)$ $\forall i = 1, \ldots, n$.

Proof

Let $\left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n}\right)$ defined on $S_1 = V \setminus U_1$ and $\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)$ defined on $S_2 = V \setminus U_2$ with $U_1, U_2 \subsetneq V$ proper subvarieties. By irreducibility $V \neq U_1 \cup U_2$, so $S_1 \cap S_2 \neq \emptyset$ and $\overline{S_1 \cap S_2} = V$.

Therefore for each $i = 1, \ldots, n$

$$f_i / g_i = p_i / q_i \text{ on } S_1 \cap S_2$$

$$\Updownarrow$$

$$f_i q_i - p_i g_i \in I(S_1 \cap S_2) = I(\overline{S_1 \cap S_2}) = I(V) \quad \square$$

Definition 10.30

Let $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ be rational mappings.
We say that the composition $\psi \circ \phi$ is defined if
$\exists p \in V$ such that ϕ is defined at p and
ψ is defined at $\phi(p) \in W$

Proposition 10.31

Let $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ be rational mappings
such that $\psi \circ \phi$ is defined.
Then $\exists U \subsetneq V$ proper subvariety such that

(i) ϕ is defined on $V \setminus U$

(ii) ψ is defined on $\phi(V \setminus U)$

(iii) $\psi \circ \phi : V \dashrightarrow W$ is a rational mapping defined on $V \setminus U$

Proof

(iii) Let $\phi = \left(\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n} \right)$ and $\psi = \left(\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m} \right)$

Then $\psi \circ \phi = (r_1, \dots, r_m)$ with

$$r_j = \frac{p_j(f_1/g_1, \dots, f_n/g_n)}{q_j(f_1/g_1, \dots, f_n/g_n)} = \frac{(g_1 \cdots g_n)^N \, p_j(f_1/g_1, \dots, f_n/g_n)}{(g_1 \cdots g_n)^N q_j(f_1/g_1, \dots, f_n/g_n)} =: \frac{P_j}{Q_j}$$

for any $N \in \mathbb{N}$. For large enough $N \in \mathbb{N}$, P_j and Q_j are polynomials and $\psi \circ \phi$ is defined on $V \setminus U$,

$$U = V(Q_1 \cdots Q_m g_1 \cdots g_n)$$

Since $\psi\circ\phi$ is defined, $\exists p \in V$ such that

ϕ defined at p: $\quad g_1(p) \cdots g_n(p) \neq 0$

ψ defined at $\phi(p)$: $\quad q_1(\phi(p)) \cdots q_m(\phi(p)) \neq 0$

$\Rightarrow Q_j(p) = (g_1(p) \cdots g_n(p))^N q_1(\phi(p)) \cdots q_m(\phi(p)) \neq 0 \quad \forall j$

so $p \in V \setminus U$ and thus $\psi\circ\phi : V \dashrightarrow W$ rational mapping.

(i) Since $V(g_1 \cdots g_n) \subset U$, ϕ defined on $V \setminus U$

(ii) Since $g_1 \cdots g_n \neq 0$ and $Q_1 \cdots Q_n \neq 0$ on $V \setminus U$, also

$$q_1(\phi(p)) \cdots q_m(\phi(p)) = \frac{Q_1(p) \cdots Q_n(p)}{(g_1(p) \cdots g_n(p))^N} \neq 0,$$

so ψ defined on $\phi(V \setminus U)$. $\square$

<u>Example 10.32</u>

Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}^3$, $\phi(t) = (t, 1/t, t^2)$ and

$$\psi : \mathbb{R}^3 \dashrightarrow \mathbb{R}, \quad \psi(x,y,z) = \frac{x+yz}{x-yz}$$

As a formal computation

$$\psi\circ\phi(t) = \psi(t, 1/t, t^2) = \frac{t + \frac{1}{t}\cdot t^2}{t - \frac{1}{t}\cdot t^2} = \frac{2t}{0}$$

which is not a rational mapping $\mathbb{R} \dashrightarrow \mathbb{R}$.

The problem is that ψ defined on $\mathbb{R}^3 \setminus V(x-yz)$, but $\phi(\mathbb{R}\setminus\{0\}) \subset V(x-yz)$, so $\nexists p \in \mathbb{R}$ s.t. ϕ defined at p and ψ defined at $\phi(p)$.

Proposition 10.33

Let $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ rational mappings and $U \subsetneq V$ subvariety such that ϕ defined on $V \setminus U$ and $W = \overline{\phi(V \setminus U)}$.

Then $\psi \circ \phi : V \dashrightarrow Z$ is a rational mapping.

Proof

By Prop 10.31, it suffices to show $\exists p \in V \setminus U$ s.t. ψ defined at $\phi(p) \in W$.

Let $Y \subsetneq W$ be a subvariety such that ψ defined on $W \setminus Y$.

By assumption $\overline{\phi(V \setminus U)} = W$, so $\phi(V \setminus U) \not\subset Y$.

Thus $\exists p \in (V \setminus U) \cap \phi^{-1}(W \setminus Y)$ $\square$

Definition 10.34

- Irreducible varieties $V \subset K^m$ and $W \subset K^n$ are birationally equivalent if $\exists \phi : V \dashrightarrow W$ and $\phi : W \dashrightarrow V$ rational mappings such that $\psi \circ \phi$ and $\phi \circ \psi$ defined with $\psi \circ \phi = id_V$ and $\phi \circ \psi = id_W$.
- A rational variety is a variety which is birationally equivalent to K^n for some $n \in \mathbb{N}$.

Lemma 10.35

Let $\phi: V \dashrightarrow W$ be a rational mapping with Zariski dense image. Then $\phi^*: K(W) \to K(V)$, $\phi^*\alpha = \alpha \circ \phi$ is a field homomorphism.

Proof

If ϕ^* is well defined, then it is a homomorphism since

$$\phi^*(\alpha + \beta) = (\alpha + \beta)\circ\phi = \alpha\circ\phi + \beta\circ\phi$$

$$\phi^*(\alpha\beta) = (\alpha\beta)\circ\phi = (\alpha\circ\phi)(\beta\circ\phi)$$

To verify ϕ^* is well defined, observe that

$$K(W) \longleftrightarrow \{\text{rational mappings } W \dashrightarrow K\}$$

$$[p]/[q],\ q \notin I(W) \longleftrightarrow \frac{p}{q} : W \setminus V(q) \to K$$

is a bijective correspondence.

Hence ϕ^* well defined by Prop 10.33.

Theorem 10.35

Irreducible varieties V and W are birationally equivalent if and only if $\exists\, \Phi: K(W) \to K(V)$ field isomorphism such that Φ is the identity on constants.

Proof

"$\Rightarrow$" Let $\phi: V \dashrightarrow W$ and $\psi: W \dashrightarrow V$ be inverse rational mappings.

This implies ϕ and ψ have Zariski dense images:

$\phi\circ\psi = id_W \Rightarrow \exists\, U \subset W$ with $\overline{U} = W$ such that

$\phi\circ\psi(a) = a\ \forall a \in U \Rightarrow U$ contained in image of ϕ

By Lemma 10.35 $\exists$ Field homomorphisms

$$\phi^*: K(W) \to K(V), \quad \phi^*\alpha = \alpha \circ \phi$$
$$\psi^*: K(V) \to K(W) \quad \psi^*\beta = \beta \circ \psi$$

On the other hand $(\phi \circ \psi)^* = (id_W)^* = id_{K(W)}$, so

$$\psi^* \circ \phi^* = (\phi \circ \psi)^* = id_{K(W)}$$

and similarly $\phi^* \circ \psi^* = id_{K(V)}$,

so $\Phi = \phi^*: K(W) \to K(V)$ is a field isomorphism

and $\Phi(c) = c \circ \phi = c$ for any constant $c \in K$.

"$\Leftarrow$" Follows by the strategy of Prop 10.8 and Thm 10.10.

(If $K[W] = K[x_1, \ldots, x_n]/I(W)$ and $K[V] = K[y_1, \ldots, y_n]/I(V)$,

define $\phi: V \dashrightarrow W$ and $\psi: W \dashrightarrow V$ by

$$\phi = (\Phi(x_1), \ldots, \Phi(x_n)), \quad \psi = (\Phi^{-1}(y_1), \ldots, \Phi^{-1}(y_n))$$

and prove they are inverses)

$\square$