Milnor's exotic structures

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Graduate student seminar

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Manifolds & smooth structures

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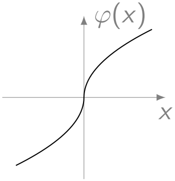
Smooth manifold: a manifold equipped with a smooth structure.

Usually the smooth structure on M is determined by an atlas $\mathcal{A}$ of charts $\varphi : U \to \mathbb{R}^n$ on open sets $U \subset M$, where the transition maps $\varphi_1 \circ \varphi_2^{-1}$ are diffeomorphisms.

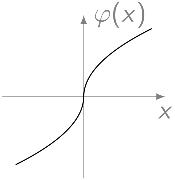
Manifolds & smooth structures

Consider $M = \mathbb{R}$ as a smooth manifold whose only chart is

$$\varphi : M \to \mathbb{R}, \quad \varphi(x) = \text{sign}(x)\sqrt{|x|}.$$



Manifolds & smooth structures



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With respect to this smooth structure $f : M \to \mathbb{R}$, $f(x) = |x|$ is smooth: $f \circ \varphi^{-1}(x) = \left|\operatorname{sign}(x)x^2\right| = x^2$

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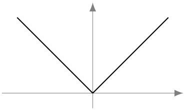

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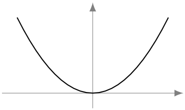

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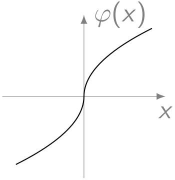

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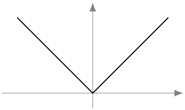

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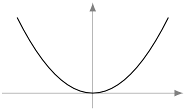

Despite this, M is diffeomorphic to standard $\mathbb{R}$.
$\varphi : M \to \mathbb{R}$ is by construction a diffeomorphism.

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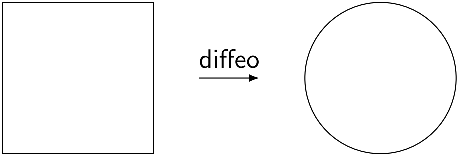
Similarly to the previous example, as a one-dimensional manifold, the square



has a smooth structure which makes it diffeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$.

On the existence & uniqueness of smooth structures

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Every manifold of dimension ≤ 3 *has a unique smooth structure.*

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- Three-dimensional manifolds have an essentially unique triangulation (Moise 1952).
- For dim ≤ 3, equivalent triangulations $\implies$ diffeomorphic smooth structures (Whitehead 1961).

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- Milnor 1956: exotic structures on the sphere S^7.
- Kervaire 1960: a 10-dimensional manifold with no smooth structure.
- Freedman 1982: a four-dimensional manifold E_8 with no smooth structure.
- Freedman 1982: existence of an exotic $\mathbb{R}^4$.
- Gompf 1985: an infinite family of exotic $\mathbb{R}^4$s.

Milnor's exotic spheres

The goal: Find a smooth manifold M that is homeomorphic, *but not diffeomorphic*, to the seven-sphere $S^7 = \{x \in \mathbb{R}^8 : \|x\| = 1\}$.

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A homeomorphism $h : S^7 \to M$ gives a new smooth structure on S^7:

$$\mathcal{A} = \{\varphi \circ h : \varphi \text{ chart of } M\}.$$

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$(S^7, \mathcal{A})$ is thus an *exotic sphere*.

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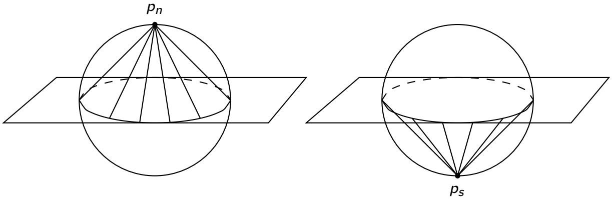
Anything diffeomorphic to S^7 would have the same invariant.
$\implies$ some manifold M_k is not diffeomorphic to S^7.

Construction of an exotic sphere

Consider two charts on S^4 given by stereographic projection with respect to the north pole and the south pole.

$$\varphi_1 : S^4 \setminus \{p_n\} \to \mathbb{R}^4 \qquad \varphi_2 : S^4 \setminus \{p_s\} \to \mathbb{R}^4$$



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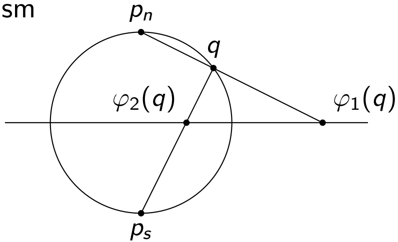
The transition map is the diffeomorphism

$$\psi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^4 \setminus \{0\}$$

$$\psi(x) = \varphi_2 \circ \varphi_1^{-1}(x) = \frac{x}{\|x\|^2}.$$



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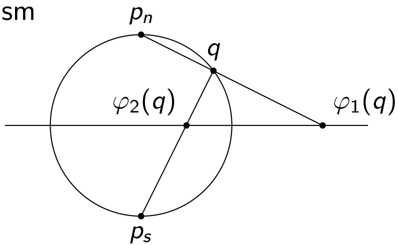
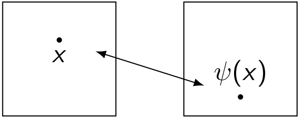


Gluing together two copies of $\mathbb{R}^4$ by identifying $x \sim \psi(x)$, i.e. $\varphi_1(q) \sim \varphi_2(q)$ gives back S^4.

$$S^4 = \mathbb{R}^4 \#_\psi \mathbb{R}^4$$



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The product $S^4 \times S^3$ would be given by

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where Φ is the diffeomorphism on $\mathbb{R}^4 \setminus \{0\} \times S^3$ given by

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But we want the manifold to be homeomorphic to S^7, so we need to modify the gluing diffeomorphism to some $\Phi(x, v) = (\psi(x), f(x, v))$.

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For any $k \in \mathbb{N}$, consider

$$f_k(x, v) = \frac{x}{\|x\|} \cdot x^k \cdot v \cdot x^{-k}$$

Construction of an exotic sphere

Let M_k be the manifold given by gluing

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This is shown by studying a function $g : M_k \to \mathbb{R}$, which in $\mathbb{R}^4 \times S^3$ is given by

$$g(x, v) = \frac{v_1}{\sqrt{1 + \|x\|^2}}.$$

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To this end, define a certain differential invariant $\lambda : \mathcal{M} \to \mathbb{Z}_7$ on the set $\mathcal{M}$ of smooth oriented 7-manifolds with a certain condition on their homology groups. $(H^3(M) = H^4(M) = 0)$

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The manifolds M_k satisfy the above conditions, and

$$\lambda(M_k) = 4k^2 + 4k \mod 7.$$

Construction of an exotic sphere

Since

$$\lambda(M_0) = 4 \cdot 0^2 + 4 \cdot 0 = 0 \mod 7 \quad \text{and}$$
$$\lambda(M_1) = 4 \cdot 1^2 + 4 \cdot 1 = 1 \mod 7,$$

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In fact M_1, which was defined by the gluing of two copies of $\mathbb{R}^4 \times S^3$ by the diffeomorphism

$$(x, v) \mapsto \left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|} \cdot x \cdot v \cdot x^{-1} \right),$$

is an exotic 7-sphere.

Further remarks on exotic structures

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Number of smooth structures:

	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathbb{R}^n$	1	1	1	∞	1	1	1	1	1	1	1	1	1
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Open problem

How many smooth structures are there on the 4-sphere?

Thank you