Quasiregular ellipticity

Def

A metric space is quasiregularly elliptic,

if $\exists$ nonconstant quasiregular $f: \mathbb{R}^n \to X$.

Recall: $f: \mathbb{R}^n \to X$ is K-quasiregular if

(i) $f \in W^{1,n}_{loc}(\mathbb{R}^n, X)$ and

(ii) $|Df(x)|^n \leq K |J_f(x)|$ for a.e. $x \in \mathbb{R}^n$

In this talk X will be a (Riemannian) manifold. [$\rightsquigarrow$ $W^{1,n}$, Df, J_f are nothing exotic]

Problem

Which spaces X can be quasiregularly elliptic?

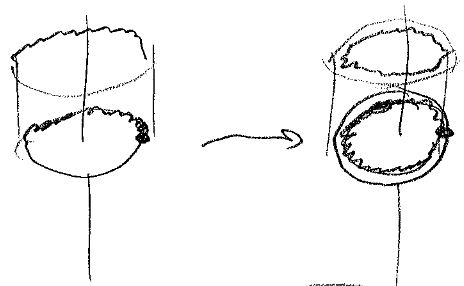
Are there topological restrictions?

QR maps are not "parametrizations", no injectivity/surjectivity required.

Examples

1) $\mathbb{R}^3 \to \mathbb{R}^3$, $(r\cos\theta, r\sin\theta, z) \mapsto (r\cos 2\theta, r\sin 2\theta, z)$

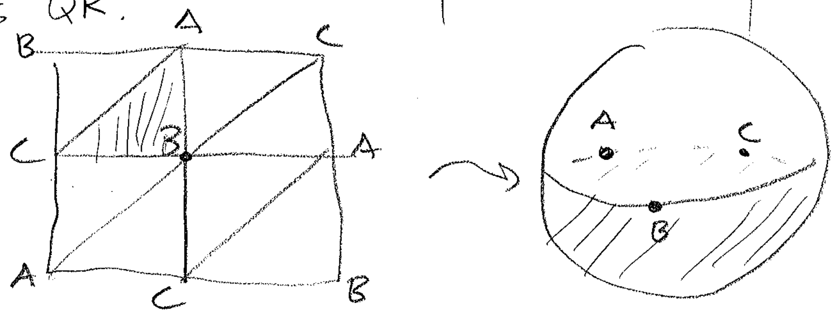
is 4-QR.



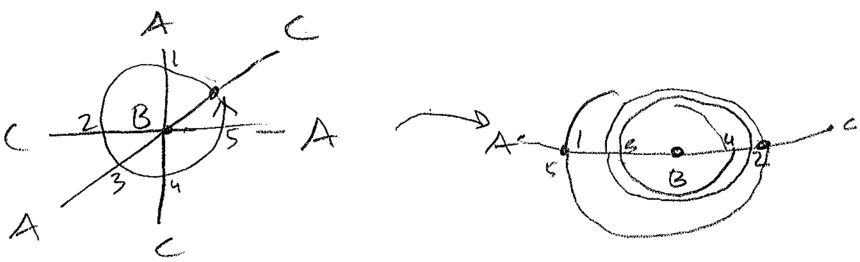
2) The Zorich map $Z: \mathbb{R}^3 \to \mathbb{R}^3$ is QR.

$Z(x,y,z) = e^z f(x,y,0)$

$Z: \mathbb{R}^2 \times \{0\} \to S^2$



QR maps are generalized branched covers.



A prototypical obstruction for QRE:

Picard's little theorem

$f: \mathbb{C} \to \mathbb{C} \setminus \{a,b\}$ analytic $\Rightarrow$ f constant

In $\mathbb{R}^2$, Stoïlow factorization

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[dr, "\substack{QC\\ \text{homeo}}"'] & \mathbb{R}^2 \setminus \{a,b\} \\ & \mathbb{R}^2 \arrow[u, "\mathbb{C}\text{-analytic}"'] \end{tikzcd}$$

$\Rightarrow$ $\mathbb{R}^2 \setminus \{a,b\}$ (or any open subset) is not QRE

In $\mathbb{R}^2$, the constant K was irrelevant

Rickman's Picard thm (1980)

$\forall K>1 \;\; \forall n \geq 2 \;\; \exists q = q(n,K)$ s.t.

$f: \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ K-QR $\Rightarrow$ f const

Rickman's Picard thm $\not\Rightarrow$ $\mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ QRE

Rickman proved $\mathbb{R}^3 \setminus \{a_1, \dots, a_q\}$ QRE (1985)

Thm (Drasin–Pankka 2015)

$\forall n \geq 3 \;\; \forall q \geq 2 \;\; \exists f: \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ QR (and surjective)

[Proof is 100 pages of essential partitions, forests, molecules, rearrangements, (skewed) Rickman partitions, pillows and pillow covers.]

[$\rightsquigarrow$ quasiregular ellipticity of subsets of $\mathbb{R}^n$ is understood]

On manifolds things are slightly more nuanced.

Def

A n-manifold M is QRE if $\exists$ Riem metric g and $\exists$ nonconstant QR $f: \mathbb{R}^n \to (M,g)$

Thm (Holopainen–Rickman 1992)

$\forall n \geq 3$ $\forall K \geq 1$ $\exists q = q(n,K)$ s.t.

$\forall$ oriented cpt diff n-mfd N $\forall$ Riem metric g on $M = N \setminus \{a_1, \dots, a_q\}$

$$f: \mathbb{R}^n \to (M,g) \ K\text{-QR} \Rightarrow f \text{ const}$$

Freedom of metric $\Rightarrow$ removal of points is not enough!

Thm (Pankka–Rajala 2011)

$S^3 \setminus S^1$ is QRE.

Naively: $\forall q$ $\mathbb{R}^3 \xrightarrow{QR} S^3 \setminus S^1 \cancel{\hookrightarrow} S^3 \setminus \{a_1, \dots, a_q\}$

QR?

The topological inclusion $S^3 \setminus S^1 \hookrightarrow S^3 \setminus \{a_1, \dots, a_q\}$ is not metrically nice,

The Riem metric blows up near S^1.

⤳ The manifold case should be studied more intrinsically, not as subsets of an ambient manifold.

⤳ Enter the world of homotopy groups and cohomology rings.

Punctured $\mathbb{R}^2$ from the topological point of view

$$\pi_1(\mathbb{R}^2) = 0$$
$$\pi_1(\mathbb{R}^2 \setminus \{a\}) = \mathbb{Z} = \mathbb{F}_1$$
$$\pi_1(\mathbb{R}^2 \setminus \{a_1, \dots, a_q\}) = \mathbb{F}_q$$

Thm (Pankka-Rajala 2011)

M connected oriented Riem n-mfd s.t. $\pi_1(M)$ has growth of order $d > n$.

$$f: \mathbb{R}^n \to N \quad QR \Rightarrow f \text{ const.}$$

[growth of order d: $\exists S \subset G$ finite s.t. $\# B_{\langle S \rangle}(r) \geq C r^d \quad \forall r \in \mathbb{N}$]

Free groups on $q \geq 2$ generators have exp growth

$\Rightarrow \mathbb{R}^2 \setminus \{a, b\}$ is not QRE

However for $n \geq 3$ $\forall$ n-mfd M

$$\pi_1(M) = 0 \Rightarrow \pi_1(M \setminus \{a_1, \dots, a_q\}) = 0$$

[Can't lasso a point!


]

[Picard thms $\rightsquigarrow$ higher homotopy groups matter]

For compact spaces, the most general result is

Thm (Bonk-Heinonen 2001)

$\forall n \geq 2$ $\forall K \geq 1$ $\exists C(n, K)$ s.t.

$\forall$ oriented cpt K-QRE n-mfd M

$$\dim H^*_{dR}(M) \leq C(n, K)$$

By quasiregular liftings, the Bonk–Heinonen thm also implies that some spaces with small cohomology are not QRE.

Example

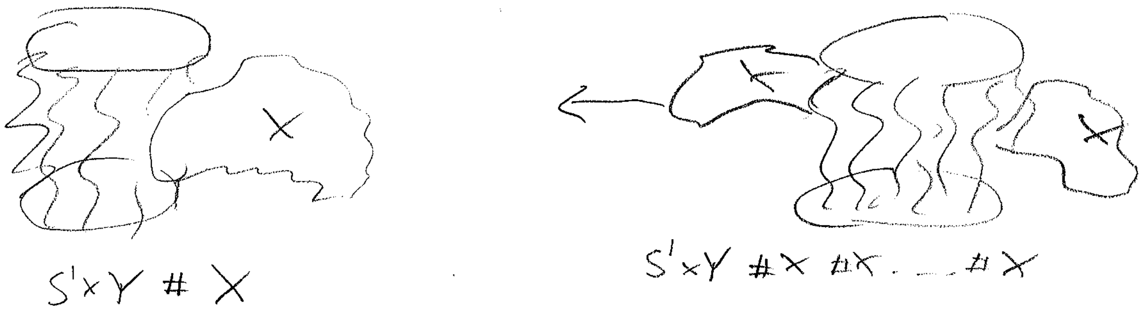
$X \# S^1 \times Y$ is not QRE,

when

X is a cpt oriented Riem n-mfd w/ non-trivial cohomology $H^k(X) \neq 0$ for some $k < n$.

Y is a cpt oriented Riem $(n-1)$-mfd

Idea: Take any r-cover $S^1 \to S^1$, $z \mapsto z^r$



Fact

$f: \mathbb{R}^n \to S^1 \times Y \# X$ QR $\Rightarrow \exists \hat{f}: \mathbb{R}^n \to S^1 \times Y \# X \# \cdots \# X$ QR, $\pi \circ \hat{f} = f$.

Now

$$\dim H^k_{dR}(S^1 \times Y \# X \# \ldots \# X) \geq r \cdot \dim H^k_{dR}(X) \to \infty \text{ as } r \to \infty.$$

$\Rightarrow$ For large enough r, $\hat{f}: \mathbb{R}^n \to S^1 \times Y \# X \ldots \# X$ is constant.

$\Rightarrow$ $f = \pi \circ \hat{f}$ is also constant.

A very, very, rough idea of the Rickman Picard construction:

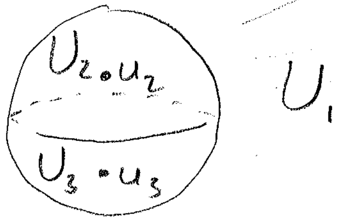
Goal: construct a QR map $f : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{a,b\}$

View $\mathbb{R}^3 \setminus \{a,b\}$ as $S^3 \setminus \{u_1, u_2, u_3)$ $(u_1 = \infty,\ u_2 = a,\ u_3 = b)$.
For simplicity take $u_2 = (0,0,\frac{1}{2})$ and $u_3 = (0,0,-\frac{1}{2})$.

Split S^3 into disjoint domains U_1, U_2, U_3 with $u_j \in \operatorname{int} U_j$.
Can take $U_1 = \mathbb{R}^3 \setminus \bar{B}(0,1)$

$U_2 = B(0,1) \cap \{x_3 > 0\}$

$U_3 = B(0,1) \cap \{x_3 < 0\}$



Now $\partial U_1 = S^2$

$\partial U_2 = (S^2 \cap \{x_3 > 0\}) \cup \underbrace{B^2}_{= \{(x_1,x_2,0):\ x_1^2 + x_2^2 \le 1\}}$

$\partial U_3 = (S^2 \cap \{x_3 < 0\}) \cup B^2$

The idea to construct $f : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{a,b\}$ is to
decompose $\mathbb{R}^3 = \overline{W_1} \cup \overline{W_2} \cup \overline{W_3}$ and define $f : \overline{W_j} \to \overline{U_j}$.

The decomposition U_1, U_2, U_3 is nice because it has perfect symmetry,
$U_1 \overset{QC}{\simeq} U_2 \setminus \{u_2\} \overset{QC}{\simeq} U_3 \setminus \{u_3\}$

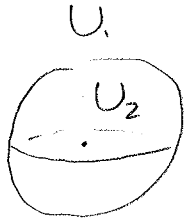
The construction of f works roughly like the construction of the Zorich map $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$.

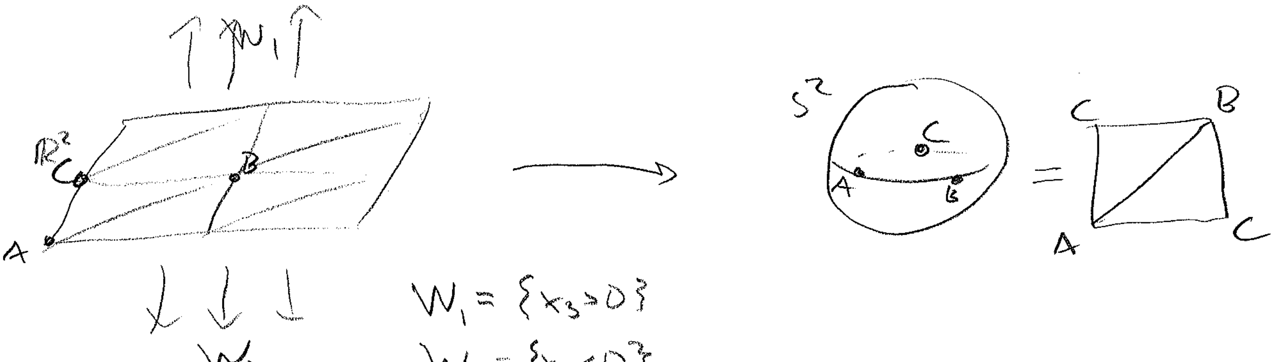
$U_1 = \mathbb{R}^3 \setminus \bar{B}(0,1)$

Zorich:
$Z(x,y,z) = e^z Z(x,y,0)$

$U_2 = B(0,1)$

$\partial U_1 = \partial U_2 = S^2$



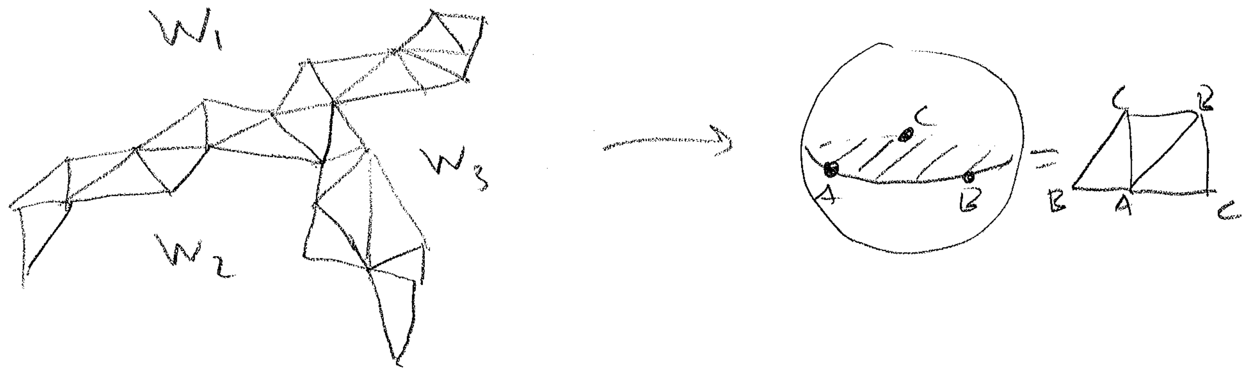


$W_1 = \{x_3 > 0\}$
$W_2 = \{x_3 < 0\}$
$\partial W_1 = \partial W_2 = \mathbb{R}^2$

In Zorich, the triangulated plane $\partial W_1 = \partial W_2 = \mathbb{R}^2$ onto the triangulated $S^2 = \partial U_1 = \partial U_2$ with a branched cover.

This branched cover is extended into the interiors of W_1, W_2 so that $\overline{W}_1 \cup \overline{W}_2 \to \overline{U}_1 \cup \overline{U}_2 \setminus \{0\}$ is QR.

In the Rickman construction, similarly want to construct a triangulated complex $P \subset \mathbb{R}^3$, $P \approx \partial W_1 \approx \partial W_2 \approx \partial W_3$



Such that

1) Each component of $\mathbb{R}^3 \setminus P$ is a topological halfspace
2) $\exists\, f: P \to S^2 \cup B^2$ branched cover that extends to a QR map $f: W_i \to U_i \setminus \{u_i\}$.

[Of course the construction of such a P is highly non-trivial]

In a vague sense, the difference between the constructions of Rickman for $n=3$ and Drasin-Pankka for $n \geq 3$ is in the method of extension and structure of P.

Rickman works with "deformation theory of 2D branched covers", whereas Drasin-Parkka impose more structure on P to ensure that each W_i is a bilipschitz half-space, making the extension easier.