MAST31005 Algebra II exercise 0 (17.01.2024)

These exercises do not count towards the course grade. The purpose is to recall relevant concepts from the prerequisite courses Algebraic structures I and II.

1. Recall the definitions of the following concepts: ring, ideal, ring homomorphism, zero divisor, field, field homomorphism, kernel of a homomorphism, irreducible polynomial.

2. Show that a subfield K of the complex numbers $\mathbb{C}$ contains the rationals $\mathbb{Q}$.

3. Let R be a ring and $I \subset R$ an ideal. Let R/I be the quotient, i.e. the set of cosets of the form $r + I$, $r \in R$.

(a) Show that the operations

$$(r + I) + (s + I) = (r + s) + I \quad \text{and} \quad (r + I) \cdot (s + I) = rs + I$$

are well defined and equip R/I with a ring structure.

(b) Show that the quotient projection

$$\pi \colon R \to R/I, \quad \pi(r) = r + I$$

is a ring homomorphism.

4. Why is the polynomial ring $\mathbb{C}[t]$ not a field?

5. Consider the finite field $\mathbb{F}_3 = \{0, 1, 2\}$ where addition and multiplication are defined modulo 3. Give an example of two polynomials $p, q \in \mathbb{F}_3[t]$ such that $p \neq q$ but $p(i) = q(i)$ for all $i \in \mathbb{F}_3$.

Recall the Rational Root Theorem: if a polynomial

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0 \in \mathbb{Z}[t], \quad a_n \neq 0$$

has a rational root $f(p/q) = 0$ with $p, q \in \mathbb{Z}$ coprime, then $p \mid a_0$ and $q \mid a_n$.

6. Let $f(t) = t^3 + 2t^2 + t + 2$.

(a) Find the rational roots of f.

(b) Give a non-trivial factorization of f in $\mathbb{Q}[t]$.

7. Let $f(t) = t^4 + t^3 + 3t^2 + 2t + 2$.

(a) Show that f has no rational roots.

(b) Is f irreducible in $\mathbb{Q}[t]$?