MAST31005 Algebra II exercise 10 (10.04.2024)

1. Let $I, J \subset K[x_1, \dots, x_n]$ be ideals. Show that if I is radical, then

$$V(I : J) = \overline{V(I) \setminus V(J)}.$$

2. Let $V, W \subset K^n$ be varieties. Show that $I(V) : I(W) = I(V \setminus W)$

3. Let $I, J \subset K[x_1, \dots, x_n]$ be ideals.

(a) Show that $I : K[x_1, \dots, x_n] = I$.

(b) Show that $J \subset I$ if and only if $I : J = K[x_1, \dots, x_n]$.

(c) Show that $J \subset \sqrt{I}$ if and only if $I : J^\infty = K[x_1, \dots, x_n]$.

(d) Give a geometric interpretation of (c) for the corresponding varieties.

4. Let $H, I, J \subset K[x_1, \dots, x_n]$ be ideals.

(a) Show that $IJ \subset H$ if and only if $I \subset H : J$.

(b) Show that $(I : J) : H = I : (JH)$.

5. Show that an ideal I is prime if and only if for any ideals J, H

$$JH \subset I \implies J \subset I \text{ or } H \subset I.$$

6. Let $p = x^2z - 6y^4 + 2xy^3z \in K[x, y, z]$. Show that there exist polynomials $q_1, q_2, q_3 \in K[x, y, z]$ such that

$$p = (x + 3) \cdot q_1 + (y - 1) \cdot q_2 + (z - 2) \cdot q_3.$$

7. Let K be a field which is not algebraically closed.

(a) Let $p \in K[x]$ be an irreducible polynomial with no roots. Show that $\langle p \rangle \subset K[x]$ is maximal.

(b) Show that if $I \subset K[x_1, \dots, x_n]$ is maximal, then $V(I)$ is empty or a singleton.

(c) Show that for each $n \geq 1$, there exists a maximal ideal $I \subset K[x_1, \dots, x_n]$ with $V(I) = \emptyset$. Hint: consider $I = \langle x_1, \dots, x_{n-1}, p(x_n) \rangle$ with p as in (a).

8. Let $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$ be a proper ideal. Show that $\sqrt{I}$ is the intersection of all maximal ideals containing I. What is the corresponding geometric statement for varieties?