MAST31005 Algebra II exercise 11 (17.04.2024)

1. Show that Theorem 9.61 implies the Weak Nullstellensatz. That is, suppose every maximal ideal $I \subset K[x_1, \ldots, x_n]$ has the form $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $a_1, \ldots, a_n \in K$ and show that if $J \subsetneq K[x_1, \ldots, x_n]$ is any proper ideal, then $V(J) \neq \emptyset$.

2. Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial. Show that $V(p)$ is an irreducible variety.

3. Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ and let $p = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ be its irreducible factorization. Show that $V(p) = V(p_1) \cup \cdots \cup V(p_m)$ is the minimal decomposition of V.

4. Let $I = \langle xz - y^3, z^3 - x^5 \rangle \subset \mathbb{Q}[x, y, z]$. Find the minimal decomposition of $V(I)$.

5. Let K be an algebraically closed field. Let $I \subset K[x_1, \ldots, x_n]$ and $q \in K[x_1, \ldots, x_n]$. Show that $V(I) \setminus V(q)$ is Zariski dense in $V(I)$ if and only if $I \colon q^\infty \subset \sqrt{I}$.

6. Let $I = \langle xy + z - 1, y^2 z^2 \rangle \subset \mathbb{C}[x, y, z]$ and $I_1 = I \cap \mathbb{C}[y, z]$ and let $\pi_1 \colon \mathbb{C}^3 \to \mathbb{C}^2$ be the projection $\pi_1(x, y, z) = (y, z)$. Show that

$$\pi_1(V(I)) = (V(z) \setminus \{(0,0)\}) \cup \{(0,1)\}.$$

7. Let $I = \langle y - xz \rangle \subset \mathbb{C}[x, y, z]$ and $V = V(I) \subset \mathbb{C}^3$. Find an explicit decomposition $\pi_1(V)$ of the form

$$\pi_1(V) = (W_1 \setminus Z_1) \cup \cdots \cup (W_m \setminus Z_m),$$

for some $m \in \mathbb{N}$, with $Z_i \subset W_i \subset \mathbb{C}^2$ varieties.