MAST31005 Algebra II exercise 4 (14.02.2024)

1. Define 'revlex' order for $\alpha, \beta \in \mathbb{N}^n$ by

$$\alpha > \beta \quad \iff \quad \text{the right-most nonzero entry of } \alpha - \beta \text{ is negative}$$

Show that revlex is not a monomial order.

2. Consider on $K[x_1, \ldots, x_8]$ the wdegrevlex order with weights $(1, 1, 2, 3, 3, 4, 4, 4)$. List all of the monomials of weighted degree 4 in increasing order.

3. Let $>_x$ be a monomial order on $K[x_1, \ldots, x_n]$ and let $>_y$ be a monomial order on $K[y_1, \ldots, y_m]$. Define the *product order* $>_{xy}$ on $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ by

$$x^\alpha y^\beta >_{xy} x^\gamma y^\delta \iff x^\alpha >_x x^\gamma \quad \text{or} \quad \begin{cases} \alpha = \gamma, \text{ and} \\ y^\beta >_y y^\delta \end{cases}$$

(a) Show that $>_{xy}$ is a monomial order.

(b) Is the lex order on $K[x, y]$ the product order of the canonical orders on $K[x]$ and $K[y]$?

4. Compute the remainder of polynomial division for $f = x^4y^2 + x^2y^2 - y + 1$ divided by the ordered tuple P with respect to the lex order on $\mathbb{Q}[x, y]$ when

(a) $P = (xy^2 - x, x - y^3)$

(b) $P = (x - y^3, xy^2 - x)$

5. Let $I = \langle x^\alpha : \alpha \in A \rangle \subset K[x_1, \ldots, x_n]$ be a monomial ideal and let $>$ be a monomial order on $K[x_1, \ldots, x_n]$. Let $\beta = \min\{\alpha : x^\alpha \in I\}$. Show that $\beta \in A$.

6. Let $I = \langle x^{\alpha_1}, \ldots, x^{\alpha_s} \rangle \subset K[x_1, \ldots, x_n]$ be a monomial ideal. Show that $f \in I$ if and only if multivariate polynomial division of f by the tuple $(x^{\alpha_1}, \ldots, x^{\alpha_s})$ gives a zero remainder.

7. Let $I = \langle x^6, x^2y^3, y^7 \rangle \subset \mathbb{Q}[x, y]$.

(a) Draw a visualization of the exponents $(m, n) \in \mathbb{N}^2$ of the monomials $x^m y^n \in I$.

(b) If we divide $f \in \mathbb{Q}[x, y]$ by the tuple (x^6, x^2y^3, y^7), which monomials can appear in the remainder?

(c) Compute the dimension of the quotient $\mathbb{Q}[x, y]/I$ as a vector space over $\mathbb{Q}$.