**MAST31005 Algebra II exercise 9 (27.03.2024)**

**1.** Let $I_1, \ldots, I_m$ and $J$ be ideals of $K[x_1, \ldots, x_n]$.

(a) Show that $(I_1 + \ldots + I_m)J = I_1 J + \ldots + I_m J$.

(b) Show that $(I_1 \cdots I_m)^r = I_1^r \cdots I_m^r$ for $r = 1, 2, 3, \ldots$

**2.** Give examples of radical ideals $I, J \subset K[x_1, \ldots, x_n]$ such that

(a) $I + J$ is not radical.

(b) $IJ$ is not radical.

**3.** Let $I, J \subset K[x_1, \ldots, x_n]$ be radical ideals. Show that $\sqrt{IJ} = I \cap J$.

**4.** Ideals $I, J \subset K[x_1, \ldots, x_n]$ are said to be *comaximal* if $I + J = K[x_1, \ldots, x_n]$.

(a) For $K = \mathbb{C}$, show that $I$ and $J$ are comaximal if and only if $V(I) \cap V(J) = \emptyset$.

(b) For some other field $K \neq \mathbb{C}$, give a counterexample to the equivalence in (a).

(c) If $I$ and $J$ are comaximal, show that $IJ = I \cap J$.

**5.** Find the Zariski closures of the following sets:

(a) The projection of the hyperbola $V(xy - 1) \subset \mathbb{R}^2$ onto the $x$-axis.

(b) The boundary of the first quadrant $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

(c) The ball $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

**6.** Let $I = \langle xy + y \rangle$ and $J = \langle xy - y \rangle$ in $\mathbb{R}[x, y]$.

(a) Determine $I \cap J = (tI + (1 - t)J) \cap \mathbb{R}[x, y]$.

(b) Show that $V(I \cap J)$ is the union of three lines $L_1, L_2, L_3$.

(c) Show that $V(tI + (1 - t)J) \subset \mathbb{R}^3$ is a union of lines $\tilde{L}_1, \tilde{L}_2$ and a plane $\tilde{L}_3$ with projections $\pi_1(\tilde{L}_i) = L_i$, $i = 1, 2, 3$, to the $xy$-plane.

(d) Let $(x, y) \in L_i$. For each $i = 1, 2, 3$, describe what information can be seen in the range of possible values for the $t$-coordinate such that $(t, x, y) \in V(tI + (1 - t)J)$.

**7.** Let $p = 40x^2 - 174xy + 189y^2$ and $q = 40x^2 + 6xy - 189y^2$. Find $h \in \mathbb{Q}[x, y]$ such that $\langle h \rangle = \langle p \rangle \cap \langle q \rangle$ and compute $\gcd(p, q) = pq/h$.