4. VARIETIES & IDEALS

Recall: For a field K,

$K[x_1,\dots,x_n]$ = polynomial ring over K in indeterminates $x_1,\dots,x_n$

Key terminology for a polynomial

$$p = \sum_\alpha a_\alpha x^\alpha \in K[x_1,\dots,x_n]$$

where $a_\alpha \in K$ nonzero for finitely many multi-indices $\alpha \in \mathbb{N}^n$:

- $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}$ a monomial
- a_α : coefficient of x^α
- $a_\alpha x^\alpha$ with $a_\alpha \neq 0$: a term of p
- $\max\{ |\alpha| = \alpha_1 + \dots + \alpha_n : a_\alpha \neq 0\} =: \deg(p)$ total degree of p

Note: sometimes no unique term of maximal degree, e.g.

$$p = \underbrace{x^2y^2 + \tfrac{1}{2}y^4}_{\text{deg } 4 \text{ terms}} + x^2 + y$$

Definition 4.1

Let K be a field and $p_1,\dots,p_s \in K[x_1,\dots,x_n]$.

The (affine) variety defined by $p_1,\dots,p_s$ is

$$V(p_1,\dots,p_s) = \{(a_1,\dots,a_n) \in K^n : p_i(a_1,\dots,a_n) = 0,\ i = 1,\dots,s\}$$

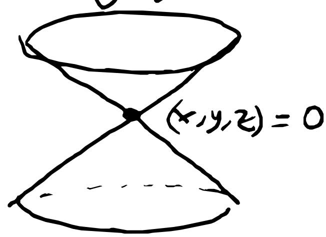
= set of solutions to $\begin{cases} p_1(x_1,\dots,x_n) = 0 \\ \quad\vdots \\ p_s(x_1,\dots,x_n) = 0 \end{cases}$

Example 4.2

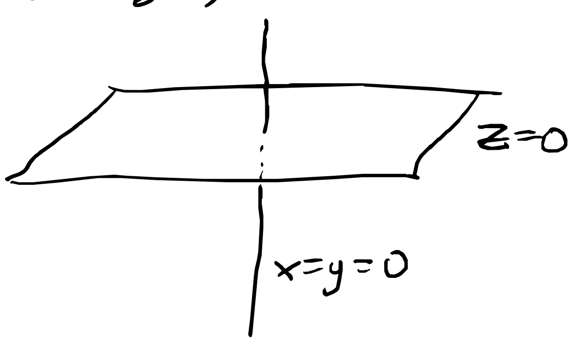
1) The circle $S = \{z \in \mathbb{C} : |z|^2 = 1\}$ is a variety over $\mathbb{R}$: if $z = x + iy$, then $|z|^2 = x^2 + y^2$, so
$$S = V(x^2 + y^2 - 1) \subset \mathbb{R}^2$$
but not over $\mathbb{C}$: $|z|^2 - 1 \notin \mathbb{C}[z]$.

2) The graph of a rational function $f \in K(t)$ is a variety. Eg. if $f = \frac{t^3 - 1}{t}$, then
$$\begin{aligned} \mathrm{Graph}(f) &= \{(t, f(t)) \in K^2 : t \in K, t \neq 0\} \\ &= V(xy - x^3 + 1) \subset K^2 \end{aligned}$$

3) Varieties may have singularities, e.g. the cone
$$V(z^2 - x^2 - y^2) \subset \mathbb{R}^3$$



4) Varieties may have different dimensional pieces:
$$V(xz, yz) \subset \mathbb{R}^3$$



Lemma 4.3

If $V, W \subset k^n$ varieties, then $V \cap W$ and $V \cup W$ varieties.

Proof

Let $V = V(p_1, \dots, p_s)$, $W = V(q_1, \dots, q_r)$.

$$V \cap W = \{a \in k^n : 0 = p_1(a) = \dots = p_s(a)\} \cap \{a \in k^n : q_1(a) = \dots q_r(a) = 0\}$$
$$= \{a \in k^n : 0 = p_1(a) = \dots = p_s(a) = q_1(a) = \dots = q_r(a) = 0\}$$
$$= V(p_1, \dots, p_s, q_1, \dots, q_r)$$

Claim: $V \cup W = V(p_i q_j : i = 1, \dots, s,\ j = 1, \dots, r)$

Proof of claim:

If $a \in V$, then $p_i(a) = 0$, $i = 1 \dots s$

$\Rightarrow (p_i q_j)(a) = p_i(a) q_j(a) = 0$, $i = 1 \dots s$, $j = 1 \dots r$

$\Rightarrow V \subset V(p_i q_j)$

Similarly $W \subset V(p_i q_j)$.

It remains to show $a \in V(p_i q_j) \Rightarrow a \in V \cup W$.

Let $a \in V(p_i q_j)$. If $a \in V$, then $a \in V \cup W$.

Otherwise $a \notin V$, so $\exists i \in \{1, \dots, s\}$ s.t. $p_i(a) \neq 0$.

However $(p_i q_j)(a) = 0$ for $j = 1, \dots, r$

$\Rightarrow q_j(a) = 0$, $j = 1, \dots, r$ $\Rightarrow$ $a \in W$. $\square$

Note: example 4.2.4 is

$$V(xz, yz) = V(z) \cup V(x, y)$$

Definition 4.4

The ideal generated by $p_1, \dots, p_s \in K[x_1, \dots, x_n]$ is

$$\langle p_1, \dots, p_s \rangle = \left\{ \sum_{i=1}^{s} q_i p_i : q_1, \dots, q_s \in K[x_1, \dots, x_n] \right\}$$

ideal $\langle p_1, \dots, p_s \rangle$ = "polynomial consequence of $p_1 = \dots = p_s = 0$"

Example 4.5

Consider a polynomial curve in $\mathbb{R}^2$

$$x = 1 + t \qquad t \in \mathbb{R}$$
$$y = -1 + t^2$$

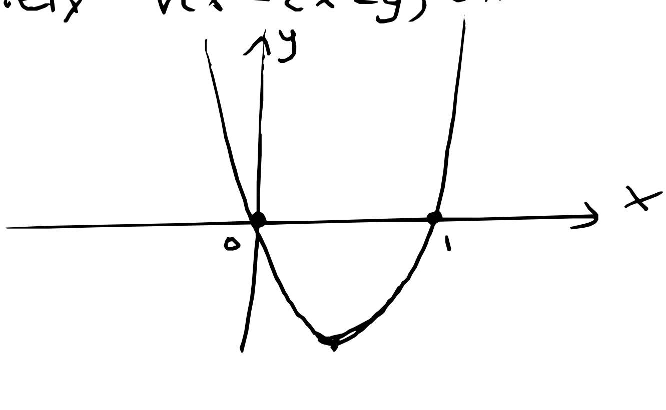
If we consider the ideal

$$I = \langle x - 1 - t,\ y + 1 - t^2 \rangle \subset \mathbb{R}[x, y, t]$$

then we find that

$$(x - 1 + t)(x - 1 - t) - (y + 1 - t^2) = (x-1)^2 - y - 1$$
$$= x^2 - 2x - y \in I$$

The original curve is a parametrization of the variety $V(x^2 - 2x - y) \subset \mathbb{R}^2$



Proposition 4.6

If $\langle p_1, \dots, p_s \rangle = \langle q_1, \dots, q_r \rangle \subset K[x_1, \dots, x_n]$, then $V(p_1, \dots, p_s) = V(q_1, \dots, q_r)$.

Proof

Let $a \in V(p_1, \dots, p_s)$, so $p_1(a) = \dots = p_s(a) = 0$.

Since $q_i \in \langle q_1, \dots, q_r \rangle = \langle p_1, \dots, p_s \rangle$

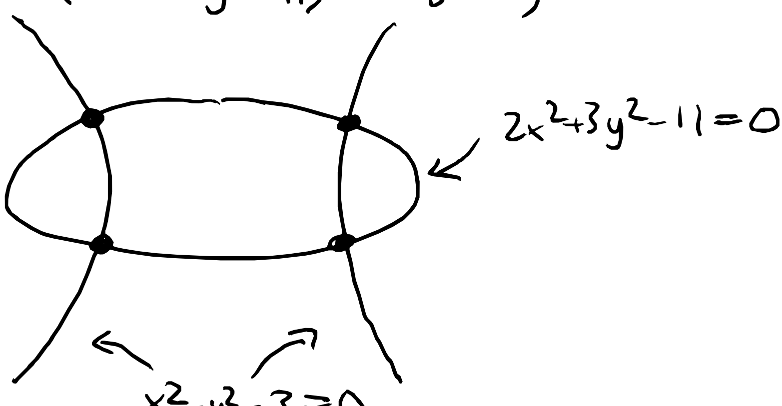
$\exists\, h_{ij} \in K[x_1, \dots, x_n]$ s.t. $q_i = \sum_{j=1}^{s} h_{ij} p_j$.

$\Rightarrow q_i(a) = \sum_j h_{ij}(a) p_j(a) = 0 \Rightarrow a \in V(q_1, \dots, q_r)$.

so $V(p_1, \dots, p_s) \subset V(q_1, \dots, q_r)$. An identical argument shows "$\supset$" $\square$.

Example 4.7

Consider $V(2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3) \subset \mathbb{R}^2$



$\langle 2x^2 + 3y^2 - 11,\ x^2 - y^2 - 3 \rangle = \langle x^2 - 4,\ y^2 - 1 \rangle$

$\Rightarrow$ intersection points are at $(\pm 2, \pm 1)$

Definition 4.8

Let $V \subset K^n$ be a variety. The ideal of V is

$$I(V) = \{ p \in K[x_1, \ldots, x_n] : p(a) = 0 \ \forall a \in V \}$$

Lemma 4.9

$I(V)$ is an ideal.

Proof

- $0 \in I(V)$ (since $0 \in K[x_1, \ldots, x_n]$ vanishes everywhere)
- if $p, q \in I(V)$, then $p+q \in I(V)$: $(p+q)(a) = p(a) + q(a) = 0$
- if $p \in I(V)$ and $f \in K[x_1, \ldots, x_n]$, then $pf \in I(V)$:

$$(pf)(a) = p(a) f(a) = 0 \cdot f(a) = 0. \quad \square$$

Example 4.10

Consider $V = V(x^2+y^2) \subset \mathbb{R}^2$. Then $V = \{(0,0)\}$.

Claim: $I(V) = \langle x, y \rangle \subset \mathbb{R}[x,y]$.

Certainly $x, y \in I(V)$ since $x(0,0) = y(0,0) = 0$.

If $p = \sum a_{nm} x^n y^m \in I(V) \subset \mathbb{R}[x,y]$ then

$$p(0,0) = a_{00} = 0$$

$$\Rightarrow p = \sum_{n>0} a_{nm} x^n y^m + \sum_{m>0} a_{0m} y^m$$

$$= \underbrace{\left(\sum_{n>0} a_{nm} x^{n-1} y^m \right)}_{\in \mathbb{R}[x,y]} x + \underbrace{\left(\sum_{m>0} a_{0m} y^{m-1} \right)}_{\in \mathbb{R}[x,y]} y \in \langle x, y \rangle$$

Proposition 4.11

Let $V, W \subset K^n$ varieties.

Then $V = W$ if and only if $I(V) = I(W)$.

and $V \subset W$ if and only if $I(V) \supset I(W)$.

Proof

By symmetry (exchange V, W) it suffices to show the latter claim.

"$\Rightarrow$" Suppose $V \subset W$ and let $p \in I(W)$.

Then $p(a) = 0 \ \forall a \in W$ so in particular $p(a) = 0 \ \forall a \in V \subset W$.

$\Rightarrow p \in I(V)$.

"$\Leftarrow$" Suppose $I(V) \supset I(W)$ and let $a \in V$.

W is a variety, so $W = V(p_1, \ldots, p_s)$, $p_1, \ldots, p_s \in K[x_1, \ldots, x_n]$.

Since $p_1(b) = \cdots = p_s(b) = 0 \ \forall b \in W$ (by definition),

$p_1, \ldots, p_s \in I(W) \subset I(V) \Rightarrow p_1(a) = \cdots = p_s(a) = 0$.

$\Rightarrow a \in W$. $\square$

We will study the relationship between V and $I(V)$ in much more detail later.

Proposition 4.12

Let $I \subset K[t]$ be an ideal.

Then $\exists\, p \in K[t]$ such that $I = \langle p \rangle$

Proof If $I = \{0\}$, take $p = 0$. Otherwise,

let $p = a_n t^n + \dots + a_0 \in I$, $a_n \neq 0$, with $\deg p$ minimal.

Since $\frac{1}{a_n} p \in I$, we may assume $a_n = 1$, so p monic.

Then $\langle p \rangle \subset I$ since I is an ideal.

Let $f \in I$. By polynomial division, we have

$$f = qp + r, \quad \deg r < \deg p.$$

Hence $r = f - qp \in I$, so minimality of $\deg p$

implies $r = 0$. $\Rightarrow$ $f \in \langle p \rangle$. $\square$

Definition 4.13

An ideal I is principal if $I = \langle p \rangle$.

Example 4.14

Not all ideals in $K[x_1, \dots, x_n]$ are principal when $n > 1$:

Consider $I = \langle x, y \rangle \subset K[x, y]$.

Suppose $I = \langle p \rangle$, $p \in K[x, y]$.

$I \neq \{0\} \Rightarrow p \neq 0$. $I \neq K[x,y] \Rightarrow \deg p \geq 1$.

Moreover $x = fp$ and $y = gp$, $f, g \in K[x, y]$

$\Rightarrow \deg f + \deg p = 1 = \deg g + \deg p$

$\Rightarrow$ $\deg f = \deg g = 0$ and $\deg p = 1$.

So $p = ax + by + c$, $a, b, c \in K$ and

$$fp = afx + bfy + cf = x$$
$$gp = agx + bgy + cg = y$$
, $f, g \neq 0$

which is impossible for all a, b, c.