Example 5.11

Polynomial division is sensitive to the monomial order: Consider the same f, p_1, p_2 as in Example 5.10 but with a different lex order: $\mathbb{Q}[t, x, y]$

$$f = x^2 - 2x - y$$

$$p_1 = -t + x - 1$$

$$p_2 = -t^2 + y + 1$$

Then none of the terms of f are divisible by $LT(p_1) = -t$ or by $LT(p_2) = -t^2$, so polynomial division gives

$$q_1 = q_2 = 0 \quad \text{and} \quad r = x^2 - 2x - y$$

$\leadsto$ polynomial division cannot always determine if $f \in \langle p_1, p_2 \rangle$

(p_1, p_2 is not a $\underbrace{\text{Gröbner basis}}_{\text{to be defined later}}$ of the ideal!)

6. MONOMIAL IDEALS

Definition 6.1

An ideal $I \subset K[x_1, \dots, x_n]$ is a monomial ideal if $\exists A \subset \mathbb{N}^n$ such that

$$I = \langle x^\alpha : \alpha \in A \rangle$$
$$= \left\{ \sum_{i=1}^{s} h_i x^{\alpha_i} : h_i \in K[x_1, \dots, x_n],\ \alpha_1, \dots, \alpha_s \in A \right\}$$

(A can be infinite)

Lemma 6.2

Let $I = \langle x^\alpha : \alpha \in A \rangle$ monomial ideal.
Then $x^\beta \in I \iff x^\alpha \mid x^\beta$ for some $\alpha \in A$

Proof

"$\Leftarrow$" If $x^\beta = x^\alpha x^\gamma$, then $x^\beta \in I$.

"$\Rightarrow$" If $x^\beta \in I$ then $x^\beta = \sum h_\alpha x^\alpha$ (sum is finite)

Write each h_α as a monomial sum

$$h_\alpha = \sum_\gamma a_{\alpha,\gamma} x^\gamma$$

so that

$$x^\beta = \sum_\alpha \sum_\gamma a_{\alpha\gamma} x^\gamma x^\alpha = \sum_\alpha \sum_\gamma a_{\alpha\gamma} x^{\alpha+\gamma}$$
$$= \sum_\delta \left(\sum_\alpha a_{\alpha, \delta-\alpha} \right) x^\delta$$

Each x^δ appearing in the sum is divisible by x^α with $\alpha \in A$.
x^β appears $\Rightarrow$ $x^\alpha \mid x^\beta$ for some $\alpha \in A$. $\square$

Lemma 6.3

Let $I \subset k[x_1, \dots, x_n]$ be a monomial ideal. Then

$$p = \sum a_\alpha x^\alpha \in I \iff x^\alpha \in I \text{ whenever } a_\alpha \neq 0$$

Proof

"$\Leftarrow$" Immediate, since an ideal contains sums

"$\Rightarrow$" Using the argument of Lemma 6.2, we deduce

$$p = \sum_\delta (c_\delta) x^\delta$$

with each x^δ appearing in the sum divisible by some x^α, $\alpha \in A$, so $x^\delta \in I$.

Since the expressions

$$\sum_\alpha a_\alpha x^\alpha = p = \sum_\delta c_\delta x^\delta$$

must be identical, we obtain $x^\alpha \in I$ when $a_\alpha \neq 0$ $\square$

Lemma 6.4

Let I, J monomial ideals. Then

$I = J$ if and only if $\{\alpha : x^\alpha \in I\} = \{\alpha : x^\alpha \in J\}$.

Proof

"$\Rightarrow$" immediate

"$\Leftarrow$" Follows directly from Lemma 6.3. $\square$

Lemma 6.2 & 6.3 give a way to visualize monomial ideals:

<u>Example 6.5</u>

Let $I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle \subset K[x,y]$,

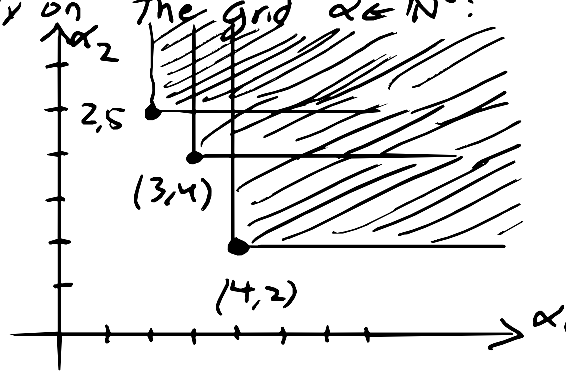
so $I = \langle x^\alpha : \alpha \in A \rangle$, $A = \{(4,2), (3,4), (2,5)\} \subset \mathbb{N}^2$

Then e.g. $x^4y^2 \mid x^\beta \Leftrightarrow \beta = (4,2) + \gamma$ for some $\gamma \in \mathbb{N}^2$

$\Longleftrightarrow \beta \in (4,2) + \mathbb{N}^2 = \{(4,2) + \gamma : \gamma \in \mathbb{N}^2\}$

Hence $x^\beta \in I \Leftrightarrow \beta \in ((4,2)+\mathbb{N}^2) \cup ((3,4)+\mathbb{N}^2) \cup ((2,5)+\mathbb{N}^2)$

Visually on the grid $\alpha \in \mathbb{N}^2$:



any polynomial with all monomials in the shaded region is in I

<u>Definition 6.6</u>

A basis of an ideal $I \subset K[x_1, \dots, x_n]$ is a subset $B \subset I$ such that $I = \langle B \rangle$

Note: There is no kind of independence assumption in Definition 6.6. (Why not? Consider $I = \langle p, q \rangle$ and solutions of $fp + gq = 0$, $f, g \in K[x_1, \dots, x_n]$)

Theorem 6.7 (Dickson's Lemma)

Let $I = \langle x^\alpha : \alpha \in A \rangle \subset K[x_1, \dots, x_n]$ monomial ideal.

Then $\exists \alpha_1, \dots, \alpha_s \in A$ s.t. $I = \langle x^{\alpha_1}, \dots, x^{\alpha_s} \rangle$

Proof (I has a finite basis)

By induction on n. For $n=1$, $A \subset \mathbb{N}$ and we may take $\alpha_1 = \min A \Rightarrow$ every x^α, $\alpha \in A$ divisible by x^{α_1}.

For $n>1$, label the indeterminates as $x_1, \dots, x_{n-1}, y$ and let

$$J = \langle x^\alpha : \alpha \in \mathbb{N}^{n-1}, \exists m \in \mathbb{N} \ x^\alpha y^m \in I \rangle$$
$$\subset K[x_1, \dots, x_{n-1}]$$

By induction $J = \langle x^{\alpha_1}, \dots, x^{\alpha_s} \rangle$

and by construction $\exists m_1, \dots, m_s \in \mathbb{N}$ s.t. $x^{\alpha_i} y^{m_i} \in I$

Let $m = \max(m_1, \dots, m_s)$ and consider

$$\tilde{I} = \langle x^{\alpha_1} y^m, \dots, x^{\alpha_s} y^m \rangle.$$

Let $x^\beta y^l \in I$ with $l \geq m$. Then $x^\beta \in J$, and thus $x^{\alpha_i} \mid x^\beta$ for some $i = 1, \dots, s$ by Lemma 6.2.

Since $l \geq m$, we obtain

$$x^{\alpha_i} y^m \mid x^\beta y^l \Rightarrow x^\beta y^l \in \tilde{I}.$$

For $l < m$ we cannot argue $x^\beta y^l \in \tilde{I}$

However, for each $l < m$ we can find suitable $\alpha_1, \dots, \alpha_s$.

For each $0 \le l < m$, define

$J_l = \langle x^\alpha : \quad x^\alpha y^l \in I \rangle < k[x_1, \dots, x_{n-1}]$

By the inductive assumption, we have

$J_l = \langle x^\alpha : \quad \alpha \in \{ \alpha_{l,1}, \dots, \alpha_{l,s_l} \} \rangle$

Denote $J_m := J$, $\alpha_{m,i} := \alpha_i$, $s_m := s$

<u>Claim</u>: $I = \langle x^\alpha y^l : (\alpha, l) \in B \rangle$ where

$B = \{ (\alpha_{l,i}, l) : 0 \le l \le m, \ 1 \le i \le s_l \}$

Proof of claim: By Lemma 6.2 & 6.4,

it suffices to show $x^\beta y^j \in I \Rightarrow x^{\alpha_{l,i}} y^l \mid x^\beta y^j$.

The case $j \ge m$ was already considered, so let $j = l < m$.

Then $x^\beta y^l \in I \Rightarrow x^\beta \in J \Rightarrow x^{\alpha_{l,i}} \mid x^\beta$

$\Rightarrow x^{\alpha_{l,i}} y^l \mid x^\beta y^l$.

In general $B \not\subset A$. However

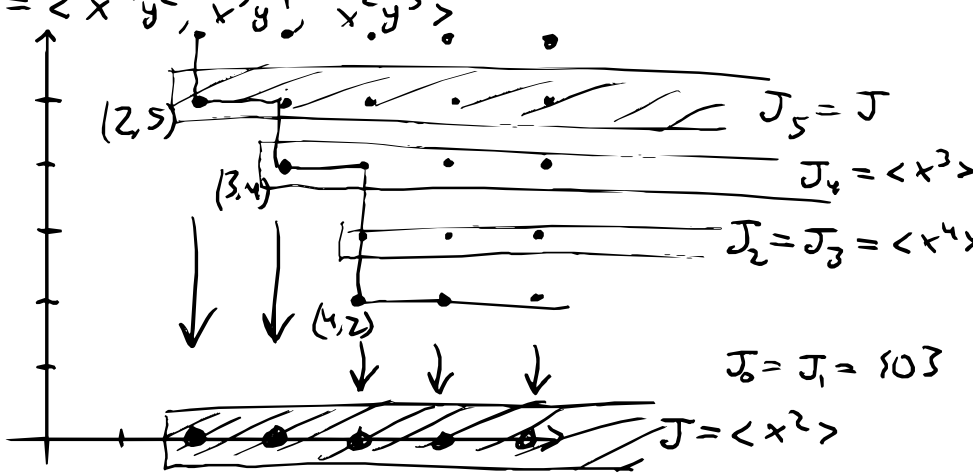
$I = \langle x^\alpha y^l : (\alpha, l) \in B \rangle = \langle x^\alpha y^l : (\alpha, l) \in A \rangle$,

so each $x^\alpha y^l$, $(\alpha, l) \in B$ divisible by $x^{\bar\alpha} y^{\bar l}$. $(\bar\alpha, \bar l) \in A$.

Then $I = \langle x^{\bar\alpha} y^{\bar l} : (\alpha, l) \in B \rangle$ $\square$

Example 6.8

$I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle$



Corollary 6.9

Let $>$ be a total order on $\mathbb{N}^n$ such that

$\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.

Then $>$ is a well order if and only if $\alpha \geq 0 \;\forall \alpha \in \mathbb{N}^n$.

Proof

"$\Rightarrow$" Let $\alpha_0 \in \mathbb{N}^n$ be the minimal element of the order.

If $\alpha_0 < 0$, then $\alpha_0 + \alpha_0 < \alpha_0$ which would contradict minimality.

"$\Leftarrow$" Let $\emptyset \neq A \subset \mathbb{N}^n$ and consider $I = \langle x^\alpha : \alpha \in A \rangle$.

Dickson's Lemma $\Rightarrow I = \langle x^{\alpha_1}, \ldots, x^{\alpha_s} \rangle$.

Let $\alpha = \min(\alpha_1, \ldots, \alpha_s)$. Then

$$\beta \in A \underset{\text{Lemma 6.2}}{\Longrightarrow} \beta = \alpha_i + \gamma \geq \alpha + \gamma \underset{\gamma \geq 0}{\geq} \alpha$$

So $\alpha = \min A$. $\square$

Proposition 6.10

Let $I \subset K[x_1,\ldots,x_n]$ a monomial ideal.

Then I has a unique minimal basis:

$$I = \langle x^{\alpha_1},\ldots,x^{\alpha_s}\rangle \quad \text{where} \quad \alpha_i \neq \alpha_j \text{ if } i \neq j.$$

Proof

By Dickson's Lemma, I has a finite basis $x^{\alpha_1},\ldots,x^{\alpha_s}$.

If $x^{\alpha_i} \mid x^{\alpha_j}$, then x^{α_j} is redundant, i.e.

$$\langle x^{\alpha_1},\ldots,x^{\alpha_s}\rangle = \langle x^{\alpha_1},\ldots,x^{\alpha_{j-1}}, x^{\alpha_{j+1}},\ldots,x_s\rangle$$

So removing redundant elements gives a minimal basis.

For uniqueness, suppose

$$I = \langle x^{\alpha_1},\ldots,x^{\alpha_s}\rangle = \langle x^{\beta_1},\ldots,x^{\beta_r}\rangle$$

with both bases minimal.

Let $i \in \{1,\ldots,s\}$. By Lemma 6.2 $\exists j \in \{1,\ldots,r\}$ s.t. $x^{\beta_j} \mid x^{\alpha_i}$

Similarly $\exists h \in \{1,\ldots,s\}$ s.t. $x^{\alpha_h} \mid x^{\beta_j}$.

Hence $x^{\alpha_h} \mid x^{\alpha_i}$ so by minimality $h = i$.

But then $x^{\alpha_i} \mid x^{\beta_j}$ & $x^{\beta_j} \mid x^{\alpha_i} \Rightarrow \alpha_i = \beta_j$

and the correspondence $i \mapsto j$ is a bijection of bases $\square$

Definition 6.11

Let $\{0\} \neq I \subset k[x_1, \ldots, x_n]$ be an ideal and $>$ a monomial order on $k[x_1, \ldots, x_n]$

- The set of leading terms of I is
$$LT(I) = \{LT(p) : p \in I\}$$
- The ideal of leading terms of I is
$$\langle LT(I) \rangle$$

Lemma 6.12

$\langle LT(I) \rangle$ is a monomial ideal and there exist $p_1, \ldots, p_s \in I$ such that $\langle LT(I) \rangle = \langle LT(p_1), \ldots, LT(p_s) \rangle$

Proof

The leading term $LT(p)$ and leading monomial $LM(p)$ only differ by a nonzero constant

$\Rightarrow$ $\langle LT(I) \rangle = \langle LM(p) : p \in I \rangle$ is a monomial ideal.

By Dickson's Lemma there is a finite subset $\{p_1, \ldots, p_s\} \subset I$ such that
$$\langle LT(I) \rangle = \langle LM(p_1), \ldots, LM(p_s) \rangle = \langle LT(p_1), \ldots, LT(p_s) \rangle \quad \square$$

Example 6.13

$I = \langle p_1, \ldots, p_s \rangle \not\Rightarrow \langle LT(I) \rangle = \langle LT(p_1), \ldots, LT(p_s) \rangle$

Consider p_1, p_2 of Example 5.11

$$p_1 = -t + x - 1$$

$$p_2 = -t^2 + y + 1$$

in lex order on $\mathbb{Q}[t, x, y]$.

Then $f = x^2 - 2x - y \in \langle p_1, p_2 \rangle$

so $LT(f) = x^2 \in \langle LT(I) \rangle$

but $\langle LT(p_1), LT(p_2) \rangle = \langle t, t^2 \rangle = \langle t \rangle$