Remark 9.71

Theorems 9.69 and 9.70 hold also for fields which are not algebraically closed, as we will later see.

THE CLOSURE THEOREM

Theorem 9.72 (The Closure Theorem)

Let K be algebraically closed and $V = V(I) \subset K^n$, and $1 \le l \le n$. Then

(i) $\overline{\pi_l(V)} = V(I_l)$ (recall $I_l = I \cap K[x_{l+1}, \dots, x_n]$)

(ii) $\exists W \subset V(I_l)$ variety such that

$$V(I_l) \setminus W \subset \pi_l(V) \subset \overline{V(I_l) \setminus W} = V(I_l)$$

Part (i) was proved in Theorem 9.39

For part (ii) we will follow a strategy similar to the proof of the Extension Theorem (Thm 8.4).

Since the variables $x_1, \dots, x_l$ and $x_{l+1}, \dots, x_n$ have different roles, relabel them as $x_1, \dots, x_l$ and $y_{l+1}, \dots, y_n$. Also denote $K[x, y] = K[x_1, \dots, x_l, y_{l+1}, \dots, y_n]$ and fix a monomial order with $x^\alpha > x^\beta \Rightarrow x^\alpha > x^\beta y^\gamma \ \forall \gamma$, (e.g. lexicographical order $x_1 > \dots > x_l > y_{l+1} > \dots > y_n$)

Proposition 9.73

Let $I \subset k[x,y]$ an ideal and $G = \{g_1, \dots, g_t\}$ be a Gröbner basis of I. For $i = 1, \dots, t$, decompose g_i as

$$g_i = c_i(y)\, x^{\alpha_i} + r_i, \quad \text{where } LT(r_i) < x^{\alpha_i},\ r_i \in k[x,y]$$

Let $b = (b_{l+1}, \dots, b_n) \in V(I_l) \subset k^{n-l}$ such that $c_i(b) \neq 0$ whenever $g_i \notin k[y]$. Then

(i) $\bar{G} = \{g_1(x,b), \dots, g_t(x,b)\} \subset k[x]$ is a Gröbner basis of $\{p(x,b) : p \in I\}$

(ii) If k algebraically closed, then $\exists a = (a_1, \dots, a_l)$ s.t. $(a,b) \in V(I)$.

Proof

(i) For $p \in k[x,y]$, denote $\bar{p} = p(x,b) \in k[x]$.

Since I is an ideal with basis G,

$\bar{I} = \{\bar{p} : p \in I\}$ is an ideal with basis $\bar{G}$.

Let $\bar{g}_i, \bar{g}_j \in \bar{G}$ nonzero, so $g_i, g_j \notin k[y]$.

Consider
$$S = c_j(y) \frac{x^{\gamma}}{x^{\alpha_i}} g_i - c_i(y) \frac{x^{\gamma}}{x^{\alpha_j}} g_j, \quad x^{\gamma} = \mathrm{lcm}(x^{\alpha_i}, x^{\alpha_j})$$

Then $LT(S) < x^{\gamma}$ and S has the standard representation

$$S = q_1 g_1 + \dots + q_t g_t, \quad LT(q_l g_l) < x^{\gamma} \text{ if } q_l g_l \neq 0$$

we obtain

$$\overbrace{c_i(b) c_j(b)}^{\neq 0} S(\bar{g}_i, \bar{g}_j) = \bar{S} = q_1(b)\bar{g}_1 + \dots + q_t(b)\bar{g}_t$$

with $LT(q_l(b)\bar{g}_l) < x^{\gamma} = \mathrm{lcm}(LT(\bar{g}_i), LT(\bar{g}_j))$

By Theorem 7.27, $\bar{G}$ is a Gröbner basis.

(ii) By construction every $\bar{g}_i \in \bar{G}$ is either zero or nonconstant, so $1 \notin \bar{I}$ since $\bar{G}$ is a Gröbner basis. By the Weak Nullstellensatz $\exists a \in V(\bar{I})$. Then $0 = \bar{g}_i(a) = g_i(a,b)$ $\forall i = 1 \to t$ so $(a,b) \in V(I)$ $\square$

Corollary 9.74

$V(I_\ell) \setminus \left(\bigcup_{g_i \notin k[y]} V(c_i) \right) \subset \pi_\ell(V)$

Proof

Let $b \in V(I_\ell) \setminus \bigcup_{g_i \notin k[y]} V(c_i)$, so $b \in V(I_\ell)$ and $c_i(b) \neq 0$ whenever $g_i \notin k[y]$.
By Proposition 9.73 $\exists a \in k^\ell$ such that $(a,b) \in V(I)$, so $b \in \pi_\ell(V)$. $\square$

Proposition 9.75

Assume k algebraically closed and G as in Prop 9.73 is reduced. If $V(I_\ell) \setminus \bigcup_{g_i \notin k[y]} V(c_i)$ is not Zariski dense in $V(I_\ell)$, then for some $i \in \{1 \to t\}$

(i) $V = V(I) = V(I + \langle c_i \rangle) \cup V(I : c_i^\infty)$

(ii) $I \subsetneq I + \langle c_i \rangle$ and $I \subsetneq I : c_i^\infty$

Proof

(i) Follows from Theorem 9.47 (i).

(ii) First observe that if $c_i \in I$, then

$LT(g_j) \mid LT(c_i)$ for some j,

$\Rightarrow LT(g_j) \mid LT(c_i) x^{\alpha_i} = LT(g_i)$.

By assumption G is reduced, so this implies $g_j = g_i$ and $LT(g_i) = LT(c_i) \in k[y]$, so $g_i \in k[y]$.

Hence whenever $g_i \notin k[y]$, $I \subsetneq I + \langle c_i \rangle$.

Denote $W = \bigcup_{g_i \notin k[y]} V(c_i) \subset k^{n-\ell}$.

By assumption $V(I_\ell) \setminus W$ is not Zariski dense in $V(I_\ell)$.

Prop 9.6.7 $\Rightarrow$ W contains some irreducible component $\tilde{V}$ of $V(I_\ell)$. Since W is a union, the irreducible component $\tilde{V}$ is contained in some $V(c_i) \supset \tilde{V}$.

Prop 9.6.7 $\Rightarrow$ $V(I_\ell) \setminus V(c_i)$ is not Zariski dense in $V(\ell)$, so

$V(I_\ell : c_i^\infty) = \overline{V(I_\ell) \setminus V(c_i)} \subsetneq V(I_\ell)$

Hence $I_\ell \subsetneq I_\ell : c_i^\infty$, so also $I \subsetneq I : c_i^\infty$,

since $p \in (I_\ell : c_i^\infty) \setminus I_\ell \Rightarrow \exists N \in \mathbb{N}$ $p c_i^N \in I_\ell \subset I$

$\Rightarrow p \in I : c_i^\infty$ but ($p \in k[y]$ is in $I \Leftrightarrow p \in I_\ell$). $\square$

Proposition 9.76

Let k be algebraically closed and $V = V(I) = V(I^1) \cup V(I^2)$

Suppose that $W_1 \subset V(I_\ell)$ and $W_2 \subset V(J_\ell)$ are such that

$$V(I_\ell^j) \setminus W_j \subset \pi_\ell(V(I^j)) \subset \overline{V(I_\ell^j) \setminus W_j}, \quad j = 1, 2$$

Then for $W = W_1 \cup W_2$

$$V(I_\ell) \setminus W \subset \pi_\ell(V) \subset \overline{V(I_\ell) \setminus W}$$

Proof

Denote $V_j = V(I^j)$. By Theorem 9.39

$$V(I_\ell) = \overline{\pi_\ell(V)} = \overline{\pi_\ell(V_1)} \cup \overline{\pi_\ell V_2)} = V(I_\ell^1) \cup V(I_\ell^2)$$

Then

$$V(I_\ell) \setminus W \subset (V(I_\ell^1) \setminus W_1) \cup (V(I_\ell^2) \setminus W_2)$$
$$\subset \pi_\ell(V_1) \cup \pi_\ell(V_2) = \pi_\ell(V)$$

proving the first inclusion.

For the second, it suffices to show $V(I_\ell) \setminus W$ is Zariski dense in $V(I_\ell)$, since $\pi_\ell(V) \subset \overline{\pi_\ell(V)} = V(I_\ell)$

By assumption $\overline{V(I_\ell^j) \setminus W_j} = \overline{\pi_\ell(I^j)} = V(I_\ell^j)$.

Prop 9.67 $\Rightarrow$ W_j does not contain any irreducible component of $V(I_\ell^j)$. Then $W = W_1 \cup W_2$ also does not contain irreducible components of $V(I_\ell) = V(I_\ell^1) \cup V(I_\ell^2)$,

Since an irreducible component of the union is an irreducible component of one (or both) of the terms.

Prop 9.67 $\Rightarrow$ $V(I_\ell) \setminus W$ is Zariski dense in $V(I_\ell)$ $\square$

Proof of the Closure Theorem (Thm 9.72(ii))

Theorem 9.63: $V = V_1 \cup \dots \cup V_m$ with V_i irreducible

Proposition 9.76: Closure Theorem for V follows if we prove the Closure Theorem for each V_i.

Thus we may assume V is irreducible.

Taking a Gröbner basis $G = \{g_1, \dots, g_t\}$ as in Prop 9.73, define $W = \bigcup_{g_i \notin k[y]} V(c_i) \subset K^{n-l}$.

Corollary 9.74 $\Rightarrow$ $V(I_l) \setminus W \subset \pi_l(V)$

Proposition 9.75 & V irreducible $\Rightarrow$ $V(I_l) \setminus W$ is Zariski dense in $V(I_l)$ $\square$

Example 9.77

In practice we don't need to precompute an irreducible decomposition

Let $I = \langle xz + y - 1, w + y + z - 2, z^2 \rangle \subset \mathbb{C}[w, x, y, z]$

Let us find $W \subset \mathbb{C}^2$ with $V(I_2) \setminus W \subset \pi_2 V(I) \subset \overline{V(I_2) \setminus W}$

A Gröbner basis in lex is

$$g_1 = w + y + z - 2 = \overbrace{1}^{c_1(y,z)} \cdot w + \overbrace{(y+z-2)}^{r_1}$$

$$g_2 = xz + y - 1 = \underbrace{z}_{c_2(y,z)} \cdot x + \underbrace{(y-1)}_{r_2}$$

$$g_3 = y^2 - 2y + 1 = (y-1)^2$$

$$g_4 = yz - z = (y-1)z$$

$$g_5 = z^2$$

Hence $V(I_2) = V(\langle g_3, g_4, g_5 \rangle) = V(y-1, z)$

We have $c_1(y,z)=1$ and $c_2(y,z)=z$

for the $g_i \notin \mathbb{C}[y,z]$, so the decomposition of

Corollary 9.74 has $W = V(1,z) = V(z)$ and

$$V(I_2) \setminus V(z) \subset \pi_2(V(I))$$

However $V(I_2) \setminus V(z) = V(y-1,z) \setminus V(z) = \emptyset$ is too small.

Apply Proposition 9.75 to split $V(I)$ into

$$V(I + \langle c_2 \rangle) \qquad \text{and} \qquad V(I : c_2^\infty)$$

We compute

$$I + \langle c_2 \rangle = \langle xz+y-1,\ w+y+z-2,\ z^2,\ z \rangle$$
$$= \langle y-1,\ w+y-2,\ z \rangle$$
$$= \langle y-1,\ w-1,\ z \rangle$$

$$I : c_2^\infty = \langle 1 \rangle, \text{ since } c_2^2 = z^2 \in I.$$

Now for $J = I + \langle c_2 \rangle$, the basis is already a Gröbner basis,

and the decomposition of Corollary 9.74 is with

$$W = V(1) = \emptyset, \quad \text{so} \quad V(J_2) = \pi_2 V(J)$$

Thus

$$\pi_2 V(I) = \pi_2 V(I + \langle c_2 \rangle) \cup \pi_2 \underbrace{V(I : c_2^\infty)}_{=\emptyset}$$
$$= V(y-1,z)$$

Theorem 9.78

Let K be algebraically closed and $V \subset K^n$ a variety.
Let $1 \le l \le n$. Then $\exists$ varieties $Z_i \subset W_i \subset K^{n-l}$, $i=1,\dots,r$

$$\pi_l(V) = W_1 \setminus Z_1 \cup \dots \cup W_r \setminus Z_r$$

Proof

If $V=\emptyset$, set $W_1 = Z_1 = \emptyset$. Otherwise define $W_1 = \overline{\pi_l(V)} \neq \emptyset$
Closure Theorem 9.72: $\exists\, Z_1 \subset W_1$ such that

$$W_1 \setminus Z_1 \subset \pi_l(V) \subset \overline{\pi_l(V)} = \overline{W_1 \setminus Z_1}$$

Set $V_1 = V \cap \pi_l^{-1}(Z_1) = \{(a_1,\dots,a_n) \in V : (a_{l+1},\dots,a_n) \in Z_1\}$.
Note: if $Z_1 = \langle p_1,\dots,p_s \rangle \subset K^{n-l}$, $p_i \in K[x_{l+1},\dots,x_n]$,
then $V_1 = V \cap \langle p_1,\dots,p_s \rangle \subset K^n$, interpreting $p_i \in K[x_1,\dots,x_n]$,
so V_1 is a variety with $V_1 \subset V$.
Moreover $V_1 \neq V$, since otherwise

$$W_1 = \overline{\pi_l(V)} = \overline{\pi_l(V_1)} \subset Z_1 \Rightarrow W_1 \setminus Z_1 = \emptyset,$$

but $\emptyset \neq \pi_l(V) \subset \overline{W_1 \setminus Z_1}$

If $a \in V \setminus V_1$, then $\pi_l(a) \notin Z_1$, so $\pi_l(a) \in W_1 \setminus Z_1$.
Thus $\pi_l(V) = (W_1 \setminus Z_1) \cup \pi_l(V_1)$

If $V_1 = \emptyset$ we are done. Otherwise, we repeat with $V = V_1$

$$\pi_l(V_1) = (W_2 \setminus Z_2) \cup \pi_l(V_2)$$

$$\Rightarrow \pi_l(V) = (W_1 \setminus Z_1) \cup (W_2 \setminus Z_2) \cup \pi_l(V_2)$$

Since the construction gives, $V \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$,
Descending chain condition $\Rightarrow \exists N : V_N = \emptyset$. $\square$