BLOWUPS AND BLOWDOWNS OF GEODESICS IN CARNOT GROUPS

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Abstract

This paper provides some partial regularity results for geodesics (i.e., isometric images of intervals) in arbitrary sub-Riemannian and sub-Finsler manifolds. Our strategy is to study infinitesimal and asymptotic properties of geodesics in Carnot groups equipped with arbitrary sub-Finsler metrics. We show that tangents of Carnot geodesics are geodesics in some groups of lower nilpotency step. Namely, every blowup curve of every geodesic in every Carnot group is still a geodesic in the group modulo its last layer. Then as a consequence we get that in every sub-Riemannian manifold any s times iterated tangent of any geodesic is a line, where s is the step of the sub-Riemannian manifold in question. With a similar approach, we also show that blowdown curves of geodesics in sub-Riemannian Carnot groups are contained in subgroups of lower rank. This latter result is also extended to rough geodesics.

Contents

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1. Introduction

In sub-Riemannian geometry, one of the major open problems is the regularity of geodesics, i.e., of isometric embeddings of intervals. Because of the presence of abnormal curves, a priori sub-Riemannian geodesics only have Lipschitz regularity, yet all known examples are C^∞. For a modern introduction to the topic we refer to [**Vit14**].

We approach the differentiability problem by considering the infinitesimal geometry, which is given by sub-Riemannian Carnot groups, and within them studying limits of dilated curves, called tangents or blowups. The main aim of this paper is to show that iterating the process of taking tangents one necessarily obtains only lines:

Theorem 1.1. *If γ is a geodesic in a sub-Riemannian manifold, then every s times iterated tangent of γ is a line, where s is the step of the sub-Riemannian manifold.*

This is a generalization of other partial results that have already been attained using a similar approach: In [**HL16**] we showed that tangents of geodesics are not corners, and in [**MPV18a**] it is shown that among all tangents at a point, one of the tangents is a line. Here "line" means a left translation of a one-parameter subgroup and "corner" means two half-lines joined together not forming a line.

A basic fact from metric geometry is that tangents of geodesics are themselves infinite geodesics. Therefore knowledge about infinite geodesics can help understand the regularity problem. For this reason, in this present work in addition to tangents we consider asymptotic cones, also called blowdowns, of infinite geodesics in Carnot groups.

Before stating our more specific results, we shall specify the notion of tangents. The notion is the same as previously used in [**HL16**, **MPV18a**]. We shall mainly restrict our considerations to Carnot groups, while allowing arbitrary length distances. For the notion of tangents within manifolds, we refer to [**MPV18b**].

Let G be a sub-Finsler Carnot group, cf. the standard definition in [**LD17**]. In G we have a Carnot-Carathéodory distance d defined by a norm on the horizontal space V_1 of G, and we have a one-parameter family of dilations, denoted by $(\delta_h)_{h>0}$. Let I be an open interval in $\mathbb{R}$, possibly $I = \mathbb{R}$. Let $\gamma : I \to G$ be a 1-Lipschitz curve and fix $\bar{t} \in I$. Denote by $\gamma_h : I_h \to G$ the curve defined on $I_h := \frac{1}{h}(I - \bar{t})$ by

$$\gamma_h(t) := \delta_{\frac{1}{h}} \left(\gamma(\bar{t})^{-1} \gamma(\bar{t} + ht) \right).$$

Notice that the last definition is just the non-abelian version of the difference quotient used in the definition of derivatives. It is trivial to check that γ_h is 1-Lipschitz and $\gamma_h(0) = 1_G$ for all $h \in (0, \infty)$. Consequently, by Ascoli-Arzelá, for every sequence $h_j \to 0$ there is a subsequence h_{j_k} and a curve $\sigma : \mathbb{R} \to G$ such that $\gamma_{h_{j_k}} \to \sigma$ uniformly on compact sets of $\mathbb{R}$. Hence, we define the collection of *tangents* as the nonempty set

$$\operatorname{Tang}(\gamma, \bar{t}) := \left\{ \sigma \mid \exists h_j \to 0 : \gamma_{h_j} \to \sigma \right\}.$$

Remark 1.2 For any Lipschitz curve $\gamma : I \to G$, the tangent cone $\operatorname{Tang}(\gamma, \bar{t})$ is a singleton if and only if γ has both one-sided derivatives at $\bar{t}$. To see the forward implication, we observe that the unique tangent must be dilation invariant and hence either a line or a corner, so the one-sided derivatives exist. For the reverse implication, we first observe that if a finite length curve has a one-sided derivative at a point, then the derivative can only be a horizontal vector. It then follows immediately from the definition of the tangent cone that there is a unique tangent curve, which is the concatenation of the two half-lines defined by the one-sided derivatives.

In the case where $\gamma : I \to M$ is a Lipschitz curve on a sub-Riemannian or sub-Finsler manifold M, we will also denote by $\operatorname{Tang}(\gamma, \bar{t})$ the collection of metric tangents of γ at $\bar{t}$. In this case, the elements of $\operatorname{Tang}(\gamma, \bar{t})$ are no longer curves in M, but instead curves in the metric tangent space $\tilde{M}$, also called the nilpotent approximation of M. We refer to [**Jea14**, Section 2.3.1] and [**MPV18b**] for details on this more general construction.

When $I = \mathbb{R}$, we will also consider limits of curves γ_{h_j} for sequences $h_j \to \infty$. Similarly to the case $h_j \to 0$, for every sequence $h_j \to \infty$ there is a subsequence h_{j_k} and a curve $\sigma : \mathbb{R} \to G$ such that $\gamma_{h_{j_k}} \to \sigma$ uniformly on compact sets of $\mathbb{R}$, so we define the collection of *asymptotic cones* as the nonempty set

$$\operatorname{Asymp}(\gamma) := \left\{ \sigma \mid \exists h_j \to \infty : \gamma_{h_j} \to \sigma \right\}.$$

The definition of $\operatorname{Asymp}(\gamma)$ is independent on the choice of $\bar{t}$ and technically the assumption that $I = \mathbb{R}$ is not necessary if we use the domains I_h as in the definition of tangents. However if I is bounded, the domains I_h degenerate to a point, and in the case where I is a half-line, all arguments are only superficially different from the line case.

Finally, we define the iterated tangent cones as the set of all tangents of (iterated) tangents, i.e., for each $k \geq 1$ we define

$$\operatorname{Tang}^{k+1}(\gamma, \bar{t}) := \bigcup_{\sigma \in \operatorname{Tang}^k(\gamma, \bar{t})} \operatorname{Tang}(\sigma, 0).$$

The elements $\sigma \in \operatorname{Tang}^k(\gamma, \bar{t})$ for any $\bar{t}$ are called *k times iterated tangents of γ*. We remark that a simple diagonal argument[1] shows that iterated tangents are also tangents, i.e., that

$$\cdots \subset \operatorname{Tang}^{k+1}(\gamma, \bar{t}) \subset \operatorname{Tang}^k(\gamma, \bar{t}) \subset \cdots \subset \operatorname{Tang}(\gamma, \bar{t}).$$

[1] If $\gamma_{h_j} \to \sigma$ and $\sigma_{k_j} \to \eta$ for some $h_j, k_j \to 0$, then for all ℓ we have $\gamma_{k_\ell h_j} \to \sigma_{k_\ell}$ and so, by a diagonal argument, there is a sequence ℓ_j such that $\gamma_{k_{\ell_j} h_j} \to \eta$.

Assume $\gamma : I \to G$ is a geodesic, i.e., $d(\gamma(a), \gamma(b)) = |a - b|$, for all $a, b \in I$. Our main results in the Carnot group setting are that every element in $\mathrm{Tang}(\gamma, \bar{t})$ is a geodesic also when projected into some quotient group of lower step, and that every element in $\mathrm{Asymp}(\gamma)$ is a geodesic inside some subgroup of lower rank (see Theorem 1.3 and Corollary 1.6, respectively).

1.1. Statement of the results. Unless otherwise stated, in what follows G will be a sub-Finsler Carnot group of nilpotency step s and $V_1 \oplus \cdots \oplus V_s = \mathfrak{g}$ will be the stratification of the Lie algebra $\mathfrak{g}$ of G. We denote by $\pi : G \to G/[G,G]$ the projection on the abelianization and by $\pi_{s-1} : G \to G/\exp(V_s)$ the projection modulo the last layer V_s of $\mathfrak{g}$.

Both groups $G/[G,G]$ and $G/\exp(V_s)$ are canonically equipped with structures of sub-Finsler Carnot groups (see Proposition 2.9). The normed vector space $G/[G,G]$ is also further canonically identified with the first layer V_1 and its dimension is the rank of G. The group $G/\exp(V_s)$ has nilpotency step $s - 1$, one lower than the original group G.

Theorem 1.3 (Blowup of geodesics)**.** *If $\gamma : I \to G$ is a geodesic and $t \in I$, then for every $\sigma \in \mathrm{Tang}(\gamma, t)$, the curve $\pi_{s-1} \circ \sigma : \mathbb{R} \to G/\exp(V_s)$ is a geodesic.*

This result implies the previously known ones from [**HL16**] that corners are not minimizing and from [**MPV18a**] that in the sub-Riemannian case one of the tangents is a line. In fact, iterating the above result, we get the following corollary.

Corollary 1.4. *If $\gamma : I \to G$ is a geodesic and $t \in I$, then for every $\sigma \in \mathrm{Tang}^{s-1}(\gamma, t)$, the horizontal projection $\pi \circ \sigma$ is a geodesic. In particular, if G is sub-Riemannian then every $\sigma \in \mathrm{Tang}^{s-1}(\gamma, t)$ is a line.*

The results of Theorem 1.3 and Corollary 1.4 also hold for a more general notion of tangents, usually called *weak tangents*, see Theorem 3.6 and Corollary 3.9. In the sub-Riemannian setting, since all infinite geodesics in step 2 are lines, Theorem 1.1 and Corollary 1.4 can be improved slightly, decreasing the number of iterations needed from s and $s - 1$ to $s - 1$ and $s - 2$, respectively.

As applications of the existence of a line tangent, we show that in every non-Abelian Carnot group where in the abelianization the infinite geodesics are lines, there is always a geodesic that loses optimality whenever it is extended (see Proposition 6.1), and show that the non-minimality of corners holds also in the non-constant rank case (see Proposition 6.2).

As mentioned in the introduction, every element in $\mathrm{Tang}(\gamma, t)$ is an infinite geodesic. We provide next other results that are valid for any infinite geodesics regardless of whether or not they are tangents.

Theorem 1.5. *If $\gamma : \mathbb{R} \to G$ is a geodesic such that $\pi \circ \gamma : \mathbb{R} \to G/[G,G]$ is not a geodesic, then there exist $R > 0$ and a hyperplane $W \subset V_1$ such that* $\mathrm{Im}(\pi \circ \gamma) \subset B_{V_1}(W, R)$.

In the above theorem, we denote by $B_{V_1}(W, R)$ the R-neighborhood of W within V_1. To prove Theorem 1.5 we shall adopt a wider viewpoint. In fact, we will consider rough geodesics and still have the same rigidity result (see Theorem 4.2).

It is possible that the claim of Theorem 1.5 could be strengthened to say that the projection of the geodesic is asymptotic to the hyperplane. In Corollary 7.20, we show that this is true for the only known family of examples of non-line infinite geodesics, arising from the explicit study of geodesics in the Engel group, see [**AS15**]. We also show that each of these geodesics is in a finite neighborhood of a line in the Engel group itself. However, by lifting the same geodesics to a step 4 Carnot group, we show that there exist infinite geodesics that are not in a finite neighborhood of any line (see Corollary 7.28).

Since in Euclidean spaces the only infinite geodesics are the straight lines, an immediate consequence of Theorem 1.5 is the following.

Corollary 1.6 (Blowdown of geodesics)**.** *If γ is a geodesic in a sub-Riemannian Carnot group $G \neq \mathbb{R}$, then there exists a proper Carnot subgroup $H < G$ containing every element of* $\mathrm{Asymp}(\gamma)$.

As with Theorem 1.5, Corollary 1.6 admits a generalization for rough geodesics (see Corollary 4.10). As a stepping stone to this generalization, we also prove that rough geodesics in Euclidean spaces have unique blowdowns (see Lemma 4.7).

Similarly as with Theorem 1.3, we can iterate Corollary 1.6 and deduce that some blowdown of an infinite geodesic in a sub-Riemannian Carnot group must be a line. We also show that in sub-Riemannian Carnot groups, every blowdown of an infinite geodesic is a line or an abnormal geodesic (see Proposition 5.5).

1.2. Organization of the paper. In Section 2 we discuss technical lemmas based on linear algebra and our error correction procedure. We introduce the concepts of minimal height and size. Proposition 2.26 is the crucial estimate and is a variant of a triangle inequality with an error term depending on the notion of size. This proposition is the key ingredient for both the proof of Theorem 1.3 and the proof of Theorem 1.5.

In Sections 3 and 4 we prove our main results. Section 3 covers our results about tangents of geodesics: Theorems 1.1 and 1.3, and Corollary 1.4, and their stronger forms for weak tangents in Theorem 3.6 and Corollary 3.9. We also give a quantified version in Theorem 3.2, which expresses the extent to which the projection of a geodesic may fail to be minimizing. Section 4 covers our results about infinite geodesics and blowdowns: Theorem 1.5 and Corollary 1.6, and their rough counterparts: Theorem 4.2 and Corollary 4.10.

In Sections 5 and 6 we discuss some applications of our main results. In Section 5 we consider infinite geodesics, and prove the statement about abnormality of blowdowns (Proposition 5.5). In Section 6 we consider applications of the existence of a line tangent. We prove the existence of non-extendable geodesics in non-abelian Carnot groups (Proposition 6.1) and the non-minimality of corners in non-constant rank sub-Riemannian manifolds (Proposition 6.2).

In Section 7 we discuss to which extent one can expect an improvement of the blowdown result Theorem 1.5, restricting our attention to rank-2 Carnot groups. In Section 7.1 we cover preliminaries on lines in Carnot groups and study when two lines are at bounded distance. In Section 7.2 we consider the example of an infinite non-line geodesic in the Engel group. We use this curve to find a counter-example to one possible strengthening of Theorem 1.5.

2. Preliminaries: minimal height, size, and error correction

2.1. Minimal height of a parallelotope and its properties.

Definition 2.1 (Minimal height of a parallelotope) Let V be a normed vector space with distance d_V. The *minimal height* of an m-tuple of points $(a_1, \dots, a_m) \in V^m$ is the smallest height of the parallelotope generated by the points, i.e.,

$$\operatorname{MinHeight}(a_1, \dots, a_m) = \min_{j \in \{1,\dots,m\}} d_V(a_j, \operatorname{span}\{a_1, \dots, \hat{a}_j, \dots, a_m\}).$$

Remarks 2.2.1 Points $a_1, \dots, a_m$ in a normed vector space are linearly independent if and only if $\operatorname{MinHeight}(a_1, \dots, a_m) \neq 0$.
2.2.2 Assume V is a Euclidean space $\mathbb{R}^r$ and denote by vol_m the usual m-dimensional volume. Let $\mathcal{P}(a_1, \dots, a_m)$ denote the parallelotope generated by the vectors $a_1, \dots, a_m$. Notice that the volume of $\mathcal{P}(a_1, \dots, a_m)$ equals the volume of any base $\mathcal{P}(a_1, \dots, \hat{a}_j, \dots, a_m)$ times the corresponding height, which is $d(a_j, \operatorname{span}\{a_1, \dots, \hat{a}_j, \dots, a_m\})$. Hence, we have

$$\begin{aligned}\operatorname{MinHeight}(a_1, \dots, a_m) &= \min_{j \in \{1,\dots,m\}} \frac{\operatorname{vol}_m \mathcal{P}(a_1, \dots, a_m)}{\operatorname{vol}_{m-1} \mathcal{P}(a_1, \dots, \hat{a}_j, \dots, a_m)} \\ &= \frac{\operatorname{vol}_m \mathcal{P}(a_1, \dots, a_m)}{\max_{j \in \{1,\dots,m\}} \operatorname{vol}_{m-1} \mathcal{P}(a_1, \dots, \hat{a}_j, \dots, a_m)}.\end{aligned}$$

Hence, if $\mathcal{P}^* := \mathcal{P}(a_1, \dots, \hat{a}_j, \dots, a_m)$ is a face of the parallelotope with maximal $(m-1)$-dimensional volume, then

$$\operatorname{MinHeight}(a_1, \dots, a_m) = \frac{\operatorname{vol}_m \mathcal{P}(a_1, \dots, a_m)}{\operatorname{vol}_{m-1} \mathcal{P}^*} = d(a_j, \operatorname{span} \mathcal{P}^*).$$

We next prove a basic lemma that uses the notion of minimal height to bound the entries of the inverse of a matrix. This bound will then be used in Lemma 2.13.

Lemma 2.3. *Let A be a matrix with columns $A_1, \dots, A_r \in \mathbb{R}^r$. If* $\mathrm{MinHeight}(A_1, \dots, A_r) > 0$*, then A is invertible and its inverse B has entries B_{kj} bounded by*

$$|B_{kj}| \leq \frac{1}{\mathrm{MinHeight}(A_1, \dots, A_r)}, \qquad \forall k, j = 1, \dots, r.$$

Proof. The fact that A is invertible follows from Remark 2.2.1. For the estimate on the entries of the inverse, we will use a well-known formula from linear algebra (see [**Lan71**, page 219]): If $A_{(k,j)}$ denotes the matrix A with row k and column j removed, then the entries of B can be calculated by

$$B_{kj} = (-1)^{k+j} \frac{\det A_{(k,j)}}{\det A}. \tag{2.4}$$

Fix $j, k \in \{1, \dots, r\}$. Let $P_k : \mathbb{R}^r \to \mathbb{R}^{r-1}$ be the projection that forgets the k-th coordinate:

$$P_k(y_1, \dots, y_r) := (y_1, \dots, \hat{y}_k, \dots, y_r).$$

Consider the following parallelotopes: Let $\mathcal{P}$ be the r-parallelotope in $\mathbb{R}^r$ determined by the points $A_1, \dots, A_r$, let $\mathcal{P}_j$ be the $(r-1)$-parallelotope in $\mathbb{R}^r$ determined by the same points excluding the vertex A_j, and let $\mathcal{P}_j^k = P_k(\mathcal{P}_j)$, which is an $(r-1)$-parallelotope in $\mathbb{R}^{r-1}$.

The geometric interpretation of the determinant states that

$$|\det A| = \mathrm{vol}_r(\mathcal{P}) \quad \text{ and } \quad \left|\det A_{(k,j)}\right| = \mathrm{vol}_{r-1}(\mathcal{P}_j^k).$$

Moreover, since $P_k(\mathcal{P}_j) = \mathcal{P}_j^k$ and the projection P_k is 1-Lipschitz, we have

$$\mathrm{vol}_{r-1}(\mathcal{P}_j^k) \leq \mathrm{vol}_{r-1}(\mathcal{P}_j).$$

By these last two observations, we have that

$$\frac{\left|\det A_{(k,j)}\right|}{|\det A|} = \frac{\mathrm{vol}_{r-1}(\mathcal{P}_j^k)}{\mathrm{vol}_r(\mathcal{P})} \leq \frac{\mathrm{vol}_{r-1}(\mathcal{P}_j)}{\mathrm{vol}_r(\mathcal{P})}. \tag{2.5}$$

Let $L_j := \mathrm{span}\{a_1, \dots, \hat{a}_j, \dots, a_m\}$. Since L_j is the span of $\mathcal{P}_j$ and $\mathcal{P}_j$ is a face of $\mathcal{P}$, we calculate the volume of $\mathcal{P}$ as in Remark 2.2.2 as

$$\mathrm{vol}_r(\mathcal{P}) = d(a_j, L_j)\,\mathrm{vol}_{r-1}(\mathcal{P}_j). \tag{2.6}$$

By the definition of MinHeight as the minimum of the distances $d(a_j, L_j)$, we conclude that

$$\begin{aligned} |B_{kj}| &\overset{(2.4)}{=} \frac{\left|\det A_{(k,j)}\right|}{|\det A|} \overset{(2.5)}{\leq} \frac{\mathrm{vol}_{r-1}(\mathcal{P}_j)}{\mathrm{vol}_r(\mathcal{P})} \\ &\overset{(2.6)}{=} \frac{1}{d(a_j, L_j)} \leq \frac{1}{\mathrm{MinHeight}(A_1, \dots, A_r)}. \end{aligned}$$

q.e.d.

Remark 2.7 For our purposes, the notion of minimal height

$$\mathrm{MinHeight}(a_1, \dots, a_r)$$

could also be replaced by the smallest singular value

$$s_{\min}(A) = \min_{\|x\|=1} \|Ax\|$$

of the matrix $A = \begin{bmatrix} a_1 & \cdots & a_r \end{bmatrix}$. The relevant use of the minimal height is through the bound of Lemma 2.3 for the entries of the inverse matrix of A. Since the operator norm $\left\|A^{-1}\right\|$ is bounded by $1/s_{\min}(A)$, we would obtain a similar estimate as in Lemma 2.3 up to a constant factor using the smallest singular value.

2.2. Size of a configuration and error correction.

Definition 2.8 A map $\pi : X \to Y$ between metric spaces is a *submetry* if $\pi(B(x, r)) = B(\pi(x), r)$ for all $x \in X$ and $r > 0$.

Proposition 2.9. *On $G/[G, G]$ and on $G/\exp(V_s)$ there are canonical structures of sub-Finsler Carnot groups such that the projections $\pi : G \to G/[G, G]$ and $\pi_{s-1} : G \to G/\exp(V_s)$ are submetries. In particular, for any $g_1, g_2 \in G$ there exists $h \in \exp(V_s)$ such that*

$$d(\pi_{s-1}(g_1), \pi_{s-1}(g_2)) = d(g_1, hg_2).$$

Proof. This proof is well known. It probably goes back to Berestovskii [**Ber89**, Theorem 1]. The key point here is that both $\exp(V_s)$ and $[G, G]$ are normal subgroups $N \triangleleft G$. Thus one can define the distance between two points $p, q \in G/N$ in the quotient as the distance

$$d_{G/N}(p, q) = d(\pi^{-1}(p), \pi^{-1}(q))$$

between their preimages under the projection $\pi : G \to G/N$. The reader can find the details in [**LR16**, Corollary 2.11].

With this definition, the resulting metric spaces are necessarily sub-Finsler Carnot groups by the characterization of [**LD15**]. Moreover for all $g_1, g_2 \in G$ we have $d_{G/N}(\pi(g_1), \pi(g_2)) \le d(g_1, g_2)$ and by local compactness there exists a point $g_2' \in \pi^{-1}(\pi(g_2))$ such that

$$d_{G/N}(\pi(g_1), \pi(g_2)) = d(g_1, g_2').$$

Since g_2 and g_2' are in the same coset, we find that $g_2' = hg_2$ for some $h \in N$. q.e.d.

Note that $G/[G, G]$ is abelian, so the canonical sub-Finsler structure on $G/[G, G]$ is the structure of a normed vector space.

Definition 2.10 (Size of a configuration) Let G be a sub-Finsler Carnot group. The *size* of an $(m+1)$-tuple of points $(g_0, \dots, g_m) \in G^{m+1}$ is

$$\mathrm{Size}(g_0, \dots, g_m) = \mathrm{MinHeight}(\pi(g_1) - \pi(g_0), \dots, \pi(g_m) - \pi(g_{m-1})). \tag{2.11}$$

Remark 2.12 Remark 2.2.1 states that non-zero MinHeight characterizes linear independence of points. Analogously, $\mathrm{Size}(g_0, \dots, g_m) \neq 0$ if and only if the horizontal projections $\pi(g_0), \dots, \pi(g_m) \in G/[G,G]$ are in general position.

The reason to consider this notion of size stems from Lemma 2.22 below, which describes our error correction procedure. Within this lemma, we need to bound the norms of solutions to a certain linear system. A convenient bound is given in Lemma 2.13 in terms of the size of a configuration of points. This dependence of the bound of solutions on the size of a configuration is the reason we are able to give restrictions on the behavior of tangents and asymptotic cones of geodesics.

Lemma 2.13 (Linear system of corrections)**.** *For every Carnot group G of rank r and step $s \geq 2$, there exists a constant $K > 0$ with the following property:*

Let $x_0, \dots, x_r \in G$ and $X_j := \log(x_{j-1}^{-1} x_j)$, for $j = 1, \dots, r$. If $\mathrm{Size}(x_0, \dots, x_r) > 0$, then for every $Z \in V_s$ there exist $Y_1, \dots, Y_r \in V_{s-1}$ such that

$$[Y_1, X_1] + \cdots + [Y_r, X_r] = Z \tag{2.14}$$

and

$$d(1_G, \exp(Y_j))^{s-1} \leq K \frac{d(1_G, \exp(Z))^s}{\mathrm{Size}(x_0, \dots, x_r)}, \qquad \forall j \in 1, \dots, r. \tag{2.15}$$

Proof. Fix arbitrary norms on the vector spaces V_{s-1} and V_s, and denote them generically as $\|\cdot\|$. Observe that the functions

$$W \mapsto d(1_G, \exp(W))^{s-1} \quad \text{and} \quad Z \mapsto d(1_G, \exp(Z))^s$$

are 1-homogeneous with respect to scalar multiplication. Therefore there exists a constant $C_1 > 1$ such that

$$\|W\| \simeq_{C_1} d(1_G, \exp(W))^{s-1} \quad \text{and} \quad \|Z\| \simeq_{C_1} d(1_G, \exp(Z))^s, \tag{2.16}$$

where $a \simeq_c b$ stands for $b/c \leq a \leq cb$.

Fix a basis $\bar{X}_1, \dots, \bar{X}_r$ of V_1. Observe that the map $(W_1, \dots, W_r) \mapsto [W_1, \bar{X}_1] + \cdots + [W_r, \bar{X}_r]$ is a linear surjection between the normed vector spaces $(V_{s-1})^r$ and V_s, where on $(V_{s-1})^r$ we use the norm $\max_{i=1,\dots,r}\{\|W_i\|\}$. Thus the map can be restricted to some subspace so that it becomes a biLipschitz linear isomorphism. In other words, there exists a constant

$C_2 > 1$ such that for all $Z \in V_s$ there exist vectors $W_1, \ldots, W_r \in V_{s-1}$ such that

$$Z = \left[W_1, \bar{X}_1\right] + \cdots + \left[W_r, \bar{X}_r\right] \tag{2.17}$$

and

$$\max_{i=1,\ldots,r} \{\|W_i\|\} \simeq_{C_2} \|Z\| \,. \tag{2.18}$$

The choice of the basis $\bar{X}_1, \ldots, \bar{X}_r$ lets us identify $G/\left[G,G\right]$ with $\mathbb{R}^r$ via the linear isomorphism $\phi : \mathbb{R}^r \to G/\left[G,G\right]$ defined by

$$\phi\left(\xi_1, \ldots, \xi_r\right) := \exp(\xi_1 \bar{X}_1 + \cdots + \xi_r \bar{X}_r + \mathfrak{g}^2),$$

where $\mathfrak{g}^2 = V_2 \oplus \cdots \oplus V_s$ and so $\exp(\mathfrak{g}^2) = [G,G]$. As a linear isomorphism, for some $C_3 > 1$, the map ϕ is a C_3-biLipschitz equivalence between $\mathbb{R}^r$ with the standard metric and $G/\left[G,G\right]$ with the quotient metric. Consequently, for all $a_1, \ldots, a_r \in \mathbb{R}^r$ we have

$$\operatorname{MinHeight}(a_1, \ldots, a_r) \simeq_{C_3} \operatorname{MinHeight}(\phi(a_1), \ldots, \phi(a_r)). \tag{2.19}$$

We now show that the constant $K := rC_1^2 C_2 C_3$ satisfies the conclusion of the lemma. Take an arbitrary $Z \in V_s$ and write it as in (2.17) for some $W_1, \ldots, W_r \in V_{s-1}$ satisfying the bound (2.18).

Given points $x_0, \ldots, x_r \in G$ with $\operatorname{Size}(x_0, \ldots, x_r) > 0$, let $v_0, \ldots, v_r \in \mathbb{R}^r$ be such that $\phi(v_j) = \pi(x_j)$ and write $v_j = (v_{j,1}, \ldots, v_{j,r})$. In other words,

$$x_j \in \exp\Big(\sum_{k=1}^{r} v_{j,k} \bar{X}_k + \mathfrak{g}^2\Big).$$

Let A be the $r \times r$ matrix whose j-th column is $A_j := v_j - v_{j-1}$, so that

$$x_{j-1}^{-1} x_j \in \exp\Big(\sum_{k=1}^{r} (v_{j,k} - v_{j-1,k}) \bar{X}_k + \mathfrak{g}^2\Big) = \exp\Big(\sum_{k=1}^{r} A_{kj} \bar{X}_k + \mathfrak{g}^2\Big). \tag{2.20}$$

The bound (2.19) combined with linearity of ϕ implies that the quantity $\operatorname{MinHeight}(A_1, \ldots, A_r)$ is comparable to $\operatorname{Size}(x_0, \ldots, x_r)$:

$$\begin{aligned}
\operatorname{MinHeight}(A_1, \ldots, A_r) &= \operatorname{MinHeight}(v_1 - v_0, \ldots, v_r - v_{r-1}) \\
&\simeq_{C_3} \operatorname{MinHeight}(\phi(v_1 - v_0), \ldots, \phi(v_r - v_{r-1})) \\
&= \operatorname{MinHeight}(\phi(v_1) - \phi(v_0), \ldots, \phi(v_r) - \phi(v_{r-1})) \\
&= \operatorname{MinHeight}(\pi(x_1) - \pi(x_0), \ldots, \pi(x_r) - \pi(x_{r-1})) \\
&= \operatorname{Size}(x_0, \ldots, x_r).
\end{aligned}$$

In particular, $\operatorname{MinHeight}(A_1, \ldots, A_r) > 0$ so we further deduce by Lemma 2.3 that A is invertible and its inverse B satisfies

$$|B_{jl}| \leq \frac{1}{\operatorname{MinHeight}(A_1, \ldots, A_r)} \leq \frac{C_3}{\operatorname{Size}(x_0, \ldots, x_r)}. \tag{2.21}$$

Set $Y_j := \sum_{l=1}^r B_{jl} W_l$. We shall verify that this choice of Y_j's satisfies the conclusion of the lemma, i.e., the properties (2.14) and (2.15).

The first property is deduced from bilinearity of the Lie bracket and the fact that AB is the identity matrix. By (2.20), we can write the vectors X_j as sums

$$X_j = \log(x_{j-1}^{-1} x_j) = \sum_{k=1}^r A_{kj} \bar{X}_k + \mathfrak{g}^2.$$

Since $\left[V_{s-1}, \mathfrak{g}^2\right] = [V_{s-1}, V_2 \oplus \cdots \oplus V_s] = 0$, it follows by bilinearity of the bracket that

$$\sum_{j=1}^r [Y_j, X_j] = \sum_{j=1}^r \left[\sum_{l=1}^r B_{jl} W_l, \sum_{k=1}^r A_{kj} \bar{X}_k\right] = \sum_{k=1}^r \sum_{l=1}^r \sum_{j=1}^r A_{kj} B_{jl} \left[W_l, \bar{X}_k\right].$$

Using the fact that AB is the identity matrix, we have $\sum_{j=1}^r A_{kj} B_{jl} = \delta_{kl}$, so the sum simplifies to

$$\sum_{j=1}^r [Y_j, X_j] = \sum_{k=1}^r \left[W_k, \bar{X}_k\right] \overset{(2.17)}{=} Z,$$

showing property (2.14).

Regarding, property (2.15), we first observe that estimating each $\|W_l\|$ by (2.18) and each $|B_{jl}|$ by (2.21), we can bound $\|Y_j\|$ by

$$\begin{aligned} \|Y_j\| = \|B_{jl} W_l\| &\leq \sum_{l=1}^r |B_{jl}| \, \|W_l\| \\ &\leq \sum_{l=1}^r \frac{C_2 C_3}{\operatorname{Size}(x_0, \ldots, x_r)} \|Z\| \\ &= \frac{r C_2 C_3}{\operatorname{Size}(x_0, \ldots, x_r)} \|Z\|. \end{aligned}$$

Then, using (2.16) to give bounds for $\|Y_j\|$ and $\|Z\|$, we conclude that

$$\begin{aligned} C_1^{-1} d(1_G, \exp(Y_j))^{s-1} \leq \|Y_j\| &\leq \frac{r C_2 C_3}{\operatorname{Size}(x_0, \ldots, x_r)} \|Z\| \\ &\leq \frac{r C_1 C_2 C_3}{\operatorname{Size}(x_0, \ldots, x_r)} d(1_G, \exp(Z))^s. \end{aligned}$$

Hence the lemma holds with the proposed constant $K = r C_1^2 C_2 C_3$. q.e.d.

As mentioned before, the following lemma describes our error correction procedure. The strategy is the same as used before in [**LM08**, **HL16**, **MPV18a**]. The geometric idea is that given a horizontal curve we perturb it adding an amount of length that depends on two factors:

(i) the desired change ($k \in G$) in the endpoint of the curve, and

(ii) the size of configuration of points $(x_0, \dots, x_r \in G)$ that the curve passes through.

However, instead of writing the argument using the language of curves, we write it as a form of a triangle inequality. The horizontal curve should be thought of as replaced by the points $x_0, \dots, x_r$ along the curve. The benefits of this approach are twofold. First, we avoid having to worry about some technicalities, such as the parametrization of the curve or the concept of inserting one curve within another. Second, a triangle-inequality form is well suited to large-scale geometry, where the local behavior of horizontal curves is irrelevant. This allows us to immediately apply our argument in the asymptotic case not only to geodesics, but to rough geodesics as well.

Lemma 2.22. *For every Carnot group G of rank r and step $s \geq 2$, there exists a constant $C > 0$ with the following property:*

Let $x_0, \dots, x_r \in G$ and $k \in \exp(V_s)$. If $\mathrm{Size}(x_0, \dots, x_r) > 0$, then

$$d(x_0, kx_r) \leq C \left(\frac{d(1_G, k)^s}{\mathrm{Size}(x_0, \dots, x_r)} \right)^{\frac{1}{s-1}} + \sum_{j=1}^{r} d(x_{j-1}, x_j).$$

Proof. Let K be the constant from Lemma 2.13 for the group G. We claim that the constant $C := 2(r+1)K^{\frac{1}{s-1}}$ will satisfy the statement of the current lemma. Given $x_0, \dots, x_r \in G$ and $k \in \exp(V_s)$, we apply Lemma 2.13 with $Z := \log(k)$ and $X_j := \log(x_{j-1}^{-1} x_j)$, for $j = 1, \dots, r$. We get the existence of $Y_1, \dots, Y_r \in V_{s-1}$ satisfying (2.14) and the bound (2.15).

Define the following points in G:

$$\begin{aligned}
& & y_j &:= \exp(Y_j), \quad \text{for } j = 1, \dots, r; \\
\alpha_0 &:= x_0, & \alpha_j &:= x_{j-1}^{-1} x_j, \quad \text{for } j = 1, \dots, r; \\
\beta_0 &:= y_1, & \beta_j &:= y_{j-1}^{-1} y_j, \quad \text{for } j = 1, \dots, r-1, \qquad \beta_r := y_r^{-1}.
\end{aligned}$$

Since $Y_j \in V_{s-1}$, by the BCH formula we have

$$\mathrm{C}_{y_j}(\alpha_j) = y_j \alpha_j y_j^{-1} = y_j \alpha_j y_j^{-1} \alpha_j^{-1} \alpha_j = \exp([Y_j, X_j]) \alpha_j,$$

where C_y denotes the conjugation by y. Consequently, since $\exp([Y_j, X_j]) \in \exp(V_s)$ commutes with everything, we have

$$\begin{aligned}
\prod_{j=0}^{r} (\alpha_j \beta_j) &= \alpha_0 \beta_0 \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_r \beta_r \\
&= \alpha_0 y_1 \alpha_1 y_1^{-1} y_2 \alpha_2 y_2^{-1} \cdots y_r \alpha_r y_r^{-1} \\
&= \alpha_0 \mathrm{C}_{y_1}(\alpha_1) \mathrm{C}_{y_2}(\alpha_2) \cdots \mathrm{C}_{y_r}(\alpha_r) \\
&= \alpha_0 \exp([Y_1, X_1]) \alpha_1 \exp([Y_2, X_2]) \alpha_2 \cdots \exp([Y_r, X_r]) \alpha_r \\
&= \exp([Y_1, X_1]) \exp([Y_2, X_2]) \cdots \exp([Y_r, X_r]) \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_r.
\end{aligned}$$

Observe that a product of exponentials is the exponential of a sum for elements in V_s and that the points α_j form the telescopic product $x_r = \alpha_0\alpha_1\alpha_2\cdots\alpha_r$. Thus the above identity simplifies to

$$\prod_{j=0}^{r}(\alpha_j\beta_j) = \exp([Y_1, X_1]+[Y_2, X_2]+\ldots+[Y_r, X_r])x_r \overset{(2.14)}{=} \exp(Z)x_r = kx_r. \tag{2.23}$$

By the definition of the points α_j for $j = 1, \ldots, r$, we have

$$d(1_G, \alpha_j) = d(x_{j-1}, x_j), \tag{2.24}$$

and for the points β_j for $j = 0, \ldots, r$, we have from (2.15) the distance estimate

$$d(1_G, \beta_j) \leq 2K^{\frac{1}{s-1}} \left(\frac{d(1_G, k)^s}{\mathrm{Size}(x_0, \ldots, x_r)}\right)^{\frac{1}{s-1}}. \tag{2.25}$$

Combining (2.23), (2.24) and (2.25) we have that

$$\begin{aligned} d(x_0, kx_r) &\overset{(2.23)}{=} d(x_0, \Pi_{j=0}^{r}(\alpha_j\beta_j)) \\ &= d(1_G, \beta_0\Pi_{j=1}^{r}(\alpha_j\beta_j)) \\ &\leq d(1_G, \beta_0) + \sum_{j=1}^{r} d(1_G, \beta_j) + \sum_{j=1}^{r} d(1_G, \alpha_j) \\ &\overset{(2.24)\&(2.25)}{\leq} 2(r+1)K^{\frac{1}{s-1}} \left(\frac{d(1_G, k)^s}{\mathrm{Size}(x_0, \ldots, x_r)}\right)^{\frac{1}{s-1}} \\ &\qquad + \sum_{j=1}^{r} d(x_{j-1}, x_j). \end{aligned}$$

Hence the lemma holds with the proposed constant $C = 2(r+1)K^{\frac{1}{s-1}}$.
q.e.d.

The following proposition contains the particular form of triangle inequality that allows us to deduce our results for both tangents and asymptotic cones of geodesics. For any set of points $x_0, \ldots, x_m \in G$ the standard triangle inequality states that

$$d(x_0, x_m) \leq \sum_{k=1}^{m} d(x_{k-1}, x_k).$$

The following proposition states that we can replace one of the terms of the sum with the distance $d(\pi_{s-1}(x_{\ell-1}), \pi_{s-1}(x_\ell))$ in the quotient group $G/\exp(V_s)$, if we pay a correction term coming from Lemma 2.22.

Theorem 1.3 for tangents will follow from the numerator of the correction term being related to the removed distance with a power $1 + \epsilon$, which implies that in the tangential limit, the correction term is irrelevant. Theorem 1.5 on the other hand will follow from the correction term being inversely related to the size of the configuration of the other points. This will allow us to apply Lemma 2.31 to constrain the behavior of geodesics on the large scale.

Proposition 2.26. *For every Carnot group G of rank r and step $s \geq 2$, there exists a constant $K > 0$ such that for any $E = (y_0, \dots, y_{r+2}) \in G^{r+3}$, $\ell \in \{1, \dots, r+2\}$ and $E_\ell := (y_0, \dots, \hat{y}_{\ell-1}, \hat{y}_\ell, \dots y_{r+2}) \in G^{r+1}$ the following modified triangle inequality holds:*

$$d(y_0, y_{r+2}) \leq d(\pi_{s-1}(y_{\ell-1}), \pi_{s-1}(y_\ell)) + K\left(\frac{d(y_{\ell-1}, y_\ell)^s}{\mathrm{Size}(E_\ell)}\right)^{\frac{1}{s-1}} + \sum_{j \neq \ell} d(y_{j-1}, y_j).$$

Proof. Since the claim of the proposition is degenerate when $\mathrm{Size}(E_\ell) = 0$, we can assume that $\mathrm{Size}(E_\ell) > 0$. Let C be the constant of Lemma 2.22 for the group G. We claim that the constant $K := 2^{\frac{s}{s-1}} C$ will satisfy the statement of the proposition.

By Proposition 2.9 there exists $h \in \exp(V_s)$ such that

$$d(y_{\ell-1}, hy_\ell) = d(\pi_{s-1}(y_{\ell-1}), \pi_{s-1}(y_\ell)). \tag{2.27}$$

We consider the points $x_j := y_j$ for $j < \ell - 1$ and $x_j := hy_{j+2}$ for $j \geq \ell - 1$. Since translation by h does not change the horizontal projection,

$$\mathrm{Size}(x_0, \dots, x_r) = \mathrm{Size}(E_\ell) > 0.$$

Applying Lemma 2.22 with $k := h^{-1}$ and the points $x_0, \dots, x_r$, we obtain the estimate

$$d(x_0, h^{-1}x_r) \leq C\left(\frac{d(1_G, h^{-1})^s}{\mathrm{Size}(x_0, \dots, x_r)}\right)^{\frac{1}{s-1}} + \sum_{j=1}^{r} d(x_{j-1}, x_j). \tag{2.28}$$

By the definition of the points x_j, for $j \neq \ell - 1$, we have

$$d(x_{j-1}, x_j) = \begin{cases} d(y_{j-1}, y_j), & \text{if } j < \ell - 1 \\ d(hy_{j+1}, hy_{j+2}), & \text{if } j > \ell - 1 \end{cases}$$

so

$$\sum_{j < \ell - 1} d(x_{j-1}, x_j) = \sum_{j < \ell - 1} d(y_{j-1}, y_j)$$

and

$$\sum_{j > \ell - 1} d(x_{j-1}, x_j) = \sum_{j > \ell + 1} d(y_{j-1}, y_j).$$

For $j = \ell - 1$ on the other hand, applying the identity (2.27) through a triangle inequality, we have

$$
\begin{aligned}
d(x_{\ell-2}, x_{\ell-1}) &= d(y_{\ell-2}, hy_{\ell+1}) \\
&\leq d(y_{\ell-2}, y_{\ell-1}) + d(y_{\ell-1}, hy_\ell) + d(hy_\ell, hy_{\ell+1}) \\
&= d(y_{\ell-2}, y_{\ell-1}) + d(\pi_{s-1}(y_{\ell-1}), \pi_{s-1}(y_\ell)) + d(y_\ell, y_{\ell+1}),
\end{aligned}
$$

filling in the missing terms $d(y_{j-1}, y_j)$ for $j = \ell - 1$ and $j = \ell + 1$. Combining the cases, we get the estimate

$$
\sum_{j=1}^{r} d(x_{j-1}, x_j) \leq d(\pi_{s-1}(y_\ell), \pi_{s-1}(y_{\ell+1})) + \sum_{j \neq \ell} d(y_{j-1}, y_j). \tag{2.29}
$$

We combine the identity (2.27) with the fact that the projection π_{s-1} is 1-Lipschitz, and we get that $d(y_{\ell-1}, hy_\ell) \leq d(y_{\ell-1}, y_\ell)$. Thus since h is in the center of G, the distance $d(1_G, h^{-1})$ can be estimated by

$$
d(1_G, h^{-1}) = d(hy_{\ell-1}, y_{\ell-1}) \leq d(hy_{\ell-1}, hy_\ell) + d(hy_\ell, y_{\ell-1}) \leq 2d(y_{\ell-1}, y_\ell). \tag{2.30}
$$

Combining (2.28) with (2.29) and (2.30) results in the desired inequality

$$
\begin{aligned}
d(y_0, y_{r+2}) \leq 2^{\frac{s}{s-1}} C \left(\frac{d(y_{\ell-1}, y_\ell)^s}{\mathrm{Size}(x_0, \ldots, x_r)} \right)^{\frac{1}{s-1}} &+ d(\pi_{s-1}(y_{\ell-1}), \pi_{s-1}(y_\ell)) \\
&+ \sum_{j \neq \ell} d(y_{j-1}, y_j).
\end{aligned}
$$

q.e.d.

2.3. Geometric lemmas about minimal height and size. None of the estimates of the rest of this section will be used for Theorem 1.3, so the reader interested in just the results about tangents can skip the following two lemmas. For the proof of Theorem 1.5 (and its generalization Theorem 4.2) we need to describe how the boundedness of the previously defined notions of Size and MinHeight relate to uniform neighborhoods of hyperplanes in the abelianization $G/[G,G]$. Lemma 2.31 describes how MinHeight and hyperplane neighborhoods are related and Lemma 2.32 gives a lower bound for Size in terms of MinHeight of a translation of the vertices.

We will only need the implications and estimates in one direction, however all of these lemmas can be generalized to include also the opposite inequalities (with possibly worse constants) and the reverse implications.

Lemma 2.31. *Let Γ be a subset of $\mathbb{R}^r$. If there exists $K > 0$ such that* $\mathrm{MinHeight}(P) \leq K$ *for all $P \in \Gamma^m$, then there exists an $(m-1)$-plane $W \subset \mathbb{R}^r$ such that $\Gamma \subset \bar{B}_{\mathbb{R}^r}(W, K)$.*

Proof. Consider first the case when Γ is a finite set. We take $P^* \in \Gamma^{m-1}$ so that the parallelotope $\mathcal{P}(P^*)$ generated by P^* maximizes the value $\mathrm{vol}_{m-1}\,\mathcal{P}(P')$ among all $P' \in \Gamma^{m-1}$. We claim that

$$\Gamma \subset \bar{B}_{\mathbb{R}^r}(\mathrm{span}(P^*), K).$$

Indeed, for every $a \in \Gamma$, since P^* has maximal volume, we have by Remark 2.2.2 that

$$d(a, \mathrm{span}(P^*)) = \frac{\mathrm{vol}_m\,\mathcal{P}(P^*, a)}{\mathrm{vol}_{m-1}\,\mathcal{P}(P^*)} \overset{2.2.2}{=} \mathrm{MinHeight}(P^*, a) \leq K.$$

Consider then the case of an infinite set Γ, and let $(p_n)_{n\in\mathbb{N}}$ be a countable dense set in Γ. Applying the lemma for the finite sets $\{p_1, \ldots, p_n\}$, we have the existence of $(m-1)$-planes W_n such that $\{p_1, \ldots, p_n\} \subset \bar{B}_{\mathbb{R}^r}(W_n, K)$. By compactness there exist an $(m-1)$-plane W and a diverging sequence n_j such that $W_{n_j} \to W$, as $j \to \infty$.

We want to prove that $\Gamma \subset \bar{B}_{\mathbb{R}^r}(W, K)$. It is enough to show that $\{p_1, \ldots, p_n\} \subset \bar{B}_{\mathbb{R}^r}(W, K+\epsilon)$, for all $n \in \mathbb{N}$ and $\epsilon > 0$. Fix such n and ϵ and fix R_n so that $\{p_1, \ldots, p_n\} \subset \bar{B}_{\mathbb{R}^r}(0, R_n)$. Then we take j large enough that $n_j > n$ and

$$\bar{B}_{\mathbb{R}^r}(W_{n_j}, K) \cap \bar{B}_{\mathbb{R}^r}(0, R_n) \subset \bar{B}_{\mathbb{R}^r}(W, K+\epsilon),$$

which is possible since $W_{n_j} \to W$, and so $\bar{B}_{\mathbb{R}^r}(W_{n_j}, K) \to \bar{B}_{\mathbb{R}^r}(W, K)$ on compact sets in the Hausdorff sense. Thus we conclude the proof of the claim:

$$\begin{aligned}\{p_1, \ldots, p_n\} \subset \{p_1, \ldots, p_{n_j}\} \cap \bar{B}_{\mathbb{R}^r}(0, R_n) &\subset \bar{B}_{\mathbb{R}^r}(W_{n_j}, K) \cap \bar{B}_{\mathbb{R}^r}(0, R_n) \\ &\subset \bar{B}_{\mathbb{R}^r}(W, K+\epsilon).\end{aligned}$$

q.e.d.

For convenience of applying Lemma 2.31 within the proof of Theorem 1.5, we give a lower bound for Size in terms of MinHeight. We will not need this bound for the proof of Theorem 1.3.

Lemma 2.32. *In any Carnot group G, there exists a constant $c > 0$ such that the following holds:*

For any $E = (g_0, \ldots, g_r) \in G^{r+1}$ and $\ell \in \{0, \ldots, r\}$, denote by $\Gamma_\ell \in (G/[G,G])^r$ the tuple of the points $\pi(g_j) - \pi(g_\ell)$, $j \neq \ell$. Then

$$\mathrm{Size}(E) \geq c \cdot \mathrm{MinHeight}(\Gamma_\ell).$$

Proof. In $\mathbb{R}^n$, consider for each $\ell \in \{0, \ldots, r\}$ the map $A^\ell : (\mathbb{R}^n)^r \to (\mathbb{R}^n)^r$, whose component functions $A^\ell_k : (\mathbb{R}^n)^r \to \mathbb{R}^n$ are defined by

$$A^\ell_k(x_1, \ldots, x_r) = \sum_{j=k}^{\ell} x_j \quad \text{for } k = 1, \ldots, \ell$$

and by

$$A^\ell_k(x_1,\dots,x_r) = \sum_{j=\ell+1}^{k} x_j \quad \text{for } k=\ell+1,\dots,r.$$

In block-matrix form, the linear map A^ℓ has the form $A^\ell = \begin{bmatrix} U & 0 \\ 0 & L \end{bmatrix}$, where

$$U = \begin{bmatrix} I & I & \dots & I \\ 0 & I & \dots & I \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} I & 0 & \dots & 0 \\ I & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \dots & I \end{bmatrix}$$

are themselves $\ell \times \ell$ and $(r-\ell)\times(r-\ell)$ upper and lower triangular block-matrices, whose $n\times n$-blocks are all either the $n\times n$ identity matrix I or zero.

From the above description, it is clear that A^ℓ is a linear bijection, so there exists a constant $C_\ell > 0$ such that A^ℓ is a C_ℓ-biLipschitz map. Thus for any set $\mathcal{P}\subset\mathbb{R}^r$, we have

$$C_\ell^{-m}\operatorname{vol}_m(\mathcal{P}) \le \operatorname{vol}_m(A^\ell(\mathcal{P})) \le C_\ell^m \operatorname{vol}_m(\mathcal{P}).$$

By the characterization of MinHeight as volume quotients in Remark 2.2.2, it follows that

$$\text{MinHeight}(A^\ell(x_1,\dots,x_r)) \le C_\ell^{2r-1}\cdot \text{MinHeight}(x_1,\dots,x_r) \tag{2.33}$$

The abelianization $G/[G,G]$ is a normed space, so there exists for some $C>0$ and $n\in\mathbb{N}$ a C-biLipschitz isomorphism $\phi : G/[G,G] \to \mathbb{R}^n$. We claim that the constant

$$c := \min_{\ell\in\{0,\dots,r\}} C^{-2} C_\ell^{1-2r} \tag{2.34}$$

satisfies the claim of the lemma.

Let $y_j := \pi(g_j) - \pi(g_{j-1})$, $j=1,\dots,r$ so that the definition (2.11) of Size is written as

$$\text{Size}(E) = \text{Size}(g_0,\dots,g_r) = \text{MinHeight}(y_1,\dots,y_r). \tag{2.35}$$

Apply the map $(\phi^{-1})^r \circ A^\ell \circ (\phi)^r : (G/[G,G])^r \to (G/[G,G])^r$ to the tuple $(y_1,\dots,y_r)\in (G/[G,G])^r$. For $k\le\ell$, we have

$$\begin{aligned}(\phi^{-1})^r \circ A^\ell_k(\phi(y_1),\dots,\phi(y_r)) &= (\phi^{-1})^r\left(\sum_{j=k}^{\ell}(\phi\circ\pi(g_j) - \phi\circ\pi(g_{j-1}))\right)\\ &= (\phi^{-1})^r\left(\phi\circ\pi(g_\ell) - \phi\circ\pi(g_{k-1})\right)\\ &= \pi(g_\ell) - \pi(g_{k-1}).\end{aligned}$$

Similarly for $k\ge \ell+1$, we have

$$(\phi^{-1})^r \circ A^\ell_k(\phi(y_1),\dots,\phi(y_r)) = \pi(g_k) - \pi(g_\ell).$$

That is, up to the sign of the elements $k \le \ell$ components, the components of $(\phi^{-1})^r \circ A^\ell(\phi(y_1), \dots, \phi(y_r))$ form exactly the tuple Γ_ℓ.

For any C-Lipschitz map f, we have

$$\mathrm{MinHeight}(f(y_1), \dots, f(y_r)) \le C \cdot \mathrm{MinHeight}(y_1, \dots, y_r).$$

Since both ϕ and ϕ^{-1} are C-Lipschitz, by (2.33) we get

$$\begin{aligned} \mathrm{MinHeight}(\Gamma_\ell) &= \mathrm{MinHeight}((\phi^{-1})^r \circ A^\ell(\phi(y_1), \dots, \phi(y_r))) \\ &\le C \cdot \mathrm{MinHeight}(A^\ell(\phi(y_1), \dots, \phi(y_r))) \\ &\overset{(2.33)}{\le} C C_\ell^{2r-1} \cdot \mathrm{MinHeight}(\phi(y_1), \dots, \phi(y_r)) \\ &\le C^2 C_\ell^{2r-1} \cdot \mathrm{MinHeight}(y_1, \dots, y_r). \end{aligned}$$

By (2.35) and (2.34) we end up with the desired estimate

$$\begin{aligned} \mathrm{Size}(E) \overset{(2.35)}{=} \mathrm{MinHeight}(y_1, \dots, y_r) &\ge \frac{1}{C^2 C_\ell^{2r-1}} \mathrm{MinHeight}(\Gamma_\ell) \\ &\overset{(2.34)}{\ge} c \cdot \mathrm{MinHeight}(\Gamma_\ell). \end{aligned}$$

q.e.d.

3. Blowups of geodesics

We next prove the results on blowups (i.e. tangents) of geodesics (Theorems 1.1 and 1.3). In fact, instead of the qualitative claim of Theorem 1.3, we will prove a slightly stronger quantified statement. We show that $\pi_{s-1} \circ \gamma$ satisfies a sublinear distance estimate on some small enough interval, implying that any tangent of $\pi_{s-1} \circ \gamma$ is a geodesic. The estimate shall follow by applying the triangle inequality of Proposition 2.26 with tuples $E = (y_0, \dots, y_{r+2})$ where only two of the points $y_{\ell-1}$ and y_ℓ will vary.

The quantified claim of Theorem 3.2 allows us to prove Theorem 1.3 and Corollary 1.4 also for the larger family of *weak tangents* (see Theorem 3.6 and Corollary 3.9), where the fixed basepoint $\bar{t}$ of the blowup is replaced with a converging sequence $t_j \to \bar{t}$. We denote by $\gamma_{h,\bar{t}} : I_{h,t} \to G$ the curve defined on $I_{h,\bar{t}} := \frac{1}{h}(I - \bar{t})$ by

$$\gamma_{h,\bar{t}}(t) := \delta_{\frac{1}{h}} \left(\gamma(\bar{t})^{-1} \gamma(\bar{t} + ht) \right). \tag{3.1}$$

The collection of all weak tangents will be denoted by

$$\mathrm{WeakTang}(\gamma, \bar{t}) := \left\{ \sigma \mid \exists h_j \to 0, t_j \to \bar{t} : \gamma_{h_j, t_j} \to \sigma \right\}$$

and the iterated weak tangent cones are denoted by

$$\mathrm{WeakTang}^{k+1}(\gamma, \bar{t}) := \bigcup_{\sigma \in \mathrm{Tang}^k(\gamma, \bar{t})} \bigcup_{t \in \mathbb{R}} \mathrm{WeakTang}(\sigma, t).$$

Theorem 3.2. *Let G be a Carnot group of step s and let $\gamma : I \to G$ be a geodesic. Then for any $\bar{t} \in I$, there exist constants $C > 0$ and $\delta > 0$ such that for all $a, b \in (\bar{t} - \delta, \bar{t} + \delta)$,*

$$|a-b| - C\,|a-b|^{\frac{s}{s-1}} \leq d(\pi_{s-1} \circ \gamma(a), \pi_{s-1} \circ \gamma(b)) \leq |a-b|\,.$$

Proof. The upper bound follows directly from the projection $\pi_{s-1} : G \to G/\exp(V_s)$ being 1-Lipschitz. The non-trivial statement is the lower bound, which will follow from Proposition 2.26.

Translating the parametrization if necessary, we may assume that $\bar{t} = 0$. Since any geodesic is still a geodesic within every Carnot subgroup containing it, we may also assume that G is the smallest Carnot subgroup containing $\gamma(I)$. Hence, if r is the rank of G, there exist $t_0, \ldots, t_r \neq 0$ such that the points $\pi \circ \gamma(t_0), \ldots, \pi \circ \gamma(t_r)$ are in general position. By Remark 2.12, we have that

$$\Delta := \mathrm{Size}(\gamma(t_0), \ldots, \gamma(t_r)) > 0. \tag{3.3}$$

Let K be the constant given by Proposition 2.26 for the Carnot group G. We claim that the constants $C := K\Delta^{-\frac{1}{s-1}}$ and $\delta := \min(|t_0|, \ldots, |t_r|)$ will satisfy the claim of the theorem.

Fix $a, b \in (-\delta, \delta)$. Consider the set of points

$$E := \{y_0, \ldots, y_{r+2}\} = \{\gamma(t_j) : \ j = 0, \ldots, r\} \cup \{\gamma(a), \gamma(b)\},$$

where the points y_j are indexed by the order in which they appear along γ. By the choice of δ, the points $\gamma(a)$ and $\gamma(b)$ are consecutive in this ordering, so there is some $\ell \in \{1, \ldots, r+2\}$ such that $y_{\ell-1} = \gamma(a)$ and $y_\ell = \gamma(b)$.

We apply Proposition 2.26 with the above E and ℓ. By (3.3), we get the estimate

$$\begin{aligned} d(y_0, y_{r+2}) \leq d(\pi_{s-1} \circ \gamma(a), \pi_{s-1} \circ \gamma(b)) + K \left(\frac{d(\gamma(a), \gamma(b))^s}{\Delta} \right)^{\frac{1}{s-1}} \\ + \sum_{j \neq \ell} d(y_{j-1}, y_j). \end{aligned} \tag{3.4}$$

By the choice of the points y_j as sequential points along the geodesic γ, we have

$$\sum_{j \neq \ell} d(y_{j-1}, y_j) = d(y_0, y_{r+2}) - d(y_{\ell-1}, y_\ell) = d(y_0, y_{r+2}) - d(\gamma(a), \gamma(b)). \tag{3.5}$$

We then apply the identity (3.5) to (3.4), we use the fact that $\gamma|_{[a,b]}$ is a geodesic, and we reorganize the terms. This gives the lower bound

$$d(\pi_{s-1} \circ \gamma(a), \pi_{s-1} \circ \gamma(b)) \geq |a-b| - K\Delta^{-\frac{1}{s-1}}\,|a-b|^{\frac{s}{s-1}}\,,$$

proving the claim of the theorem. q.e.d.

Theorem 1.3 shall follow immediately from Theorem 3.2 by taking any limit of dilations $h_k \to 0$. We prove the result for the larger family of all weak tangents.

Theorem 3.6. *If $\gamma : I \to G$ is a geodesic and $t \in I$, then for every $\sigma \in \mathrm{WeakTang}(\gamma, t)$, the curve $\pi_{s-1} \circ \sigma : \mathbb{R} \to G/\exp(V_s)$ is a geodesic.*

Proof. Reparametrizing and left-translating if necessary, we may assume that $t = 0$ and $\gamma(0) = 1_G$. Then $\sigma \in \mathrm{WeakTang}(\gamma, 0)$ is given by some sequences $h_k \to 0$ and $t_k \to 0$ as $\sigma = \lim_{k\to\infty} \gamma_{h_k,t_k}$.

For any $h > 0$, $t \in I$, and $a, b \in I_{h,t}$, expanding the definition (3.1) of the dilated curve $\gamma_{h,t}$, we obtain the identity

$$d(\gamma_{h,t}(a), \gamma_{h,t}(b)) = \frac{1}{h} d(\gamma(ha), \gamma(hb)). \tag{3.7}$$

Let $C > 0$ and $\delta > 0$ be the constants of Theorem 3.2 for the geodesic γ. Rephrasing the statement of Theorem 3.2 for $\gamma_{h,t}$ using (3.7), we get for all $a, b \in ((-\delta - t)/h, (\delta - t)/h)$ that

$$|a - b| - Ch^{\frac{1}{s-1}} |a - b|^{\frac{s}{s-1}} \leq d(\pi_{s-1} \circ \gamma_{h,t}(a), \pi_{s-1} \circ \gamma_{h,t}(b)) \leq |a - b| . \tag{3.8}$$

For any $a, b \in \mathbb{R}$, the condition $a, b \in ((-\delta - t_k)/h_k, (\delta - t_k)/h_k)$ is satisfied for any large enough indices $k \in \mathbb{N}$. Thus taking the limit of (3.8) as $h = h_k \to 0$, we get for the limit curve $\pi_{s-1} \circ \sigma = \lim_{k\to\infty} \pi_{s-1} \circ \gamma_{h_k,t_k}$ the estimate

$$|a - b| \leq d(\pi_{s-1} \circ \sigma(a), \pi_{s-1} \circ \sigma(b)) \leq |a - b| ,$$

showing that $\pi_{s-1} \circ \sigma$ is a geodesic. q.e.d.

By induction from Theorem 3.6 we obtain the following corollary, which is the stronger form of Corollary 1.4.

Corollary 3.9. *If $\gamma : I \to G$ is a geodesic and $t \in I$, then for every $\sigma \in \mathrm{WeakTang}^{s-1}(\gamma, t)$, the horizontal projection $\pi \circ \sigma$ is a geodesic. In particular, if G is sub-Riemannian then every $\sigma \in \mathrm{WeakTang}^{s-1}(\gamma, t)$ is a line.*

We prove next that Theorem 1.1 follows from the fact that the metric tangent of a sub-Riemannian manifold is a quotient of a sub-Riemannian Carnot group, which is a well known theorem attributed to Bellaïche [**Bel96**].

Proof of Theorem 1.1. Let M be a sub-Riemannian manifold. Let s be the step of the sub-Riemannian manifold, i.e., Lie brackets of length s of the horizontal vector fields in M span the tangent spaces T_pM at each point $p \in M$. Let $\gamma : I \to M$ be a geodesic.

Following [**MPV18b**, Theorem 3.6], we see that any metric tangent σ of γ is a geodesic in the nilpotent approximation $\tilde{M}$ of M. By [**Jea14**, Theorem 2.7], the nilpotent approximation of a sub-Riemannian manifold is a homogeneous space G/H, where G is a Carnot group of step s and $H < G$ is a closed dilation invariant Lie subgroup. In particular, any iterated tangent of σ gives another geodesic in $\tilde{M} = G/H$.

On the other hand, since the projection $\pi : G \to G/H$ is a submetry, the geodesic σ can be lifted to a geodesic $\tilde{\sigma}$ in G. Applying Corollary 1.4 we see that any $s-1$ times iterated tangent of $\tilde{\sigma}$ is a line. Projecting back to G/H, we see that also necessarily any $s-1$ times iterated of σ must be a line. Since σ was an arbitrary tangent of γ, it follows that any s times iterated tangent of γ is a line. q.e.d.

4. Blowdowns of rough geodesics

In this section we prove Theorem 1.5 and Corollary 1.6. Due to our formulation of the core of the argument (Proposition 2.26) as a triangle inequality, we are able to prove the stronger claims of Theorem 4.2 and Corollary 4.10 for rough geodesics.

To make the terminology precise, by *rough geodesic*, we mean a not-necessarily-continuous curve that is a $(1, C)$-quasi-geodesic for some $C \geq 0$. By a $(1, C)$-*quasi-geodesic* we mean a $(1, C)$-*quasi-isometric embedding*, i.e., some $\gamma : I \to G$ such that

$$|t_1 - t_2| - C \leq d(\gamma(t_1), \gamma(t_2)) \leq |t_1 - t_2| + C, \qquad \forall t_1, t_2 \in I. \tag{4.1}$$

Thus a $(1, 0)$-quasi-geodesic is exactly a geodesic.

Theorem 4.2. *If $\gamma : \mathbb{R} \to G$ is a $(1, C)$-quasi-geodesic, then one of the following holds:*

(4.2.i) There exist a hyperplane $W \subset V_1$ and some $R > 0$ such that $\mathrm{Im}(\pi \circ \gamma) \subset B_{V_1}(W, R)$.
(4.2.ii) There exists $C' \geq 0$ such that $\pi \circ \gamma : \mathbb{R} \to G/[G, G]$ is a $(1, C')$-quasi-geodesic.

Moreover, one can take $C' = (r + 2)^{s-1} C$, where r is the rank and s is the step of the Carnot group G.

Proof. Assume (4.2.i) does not hold. We claim that it is enough to show that $\pi_{s-1} \circ \gamma$ is a $(1, C_1)$-quasi-geodesic with $C_1 := (r + 2)C$. Indeed, then we can iterate: the curve $\pi_{s-1} \circ \gamma$ has the same projection as γ on $G/[G, G]$. Thus, (4.2.i) does not hold for $\pi_{s-1} \circ \gamma$ either, and we have that $\pi_{s-2} \circ \pi_{s-1} \circ \gamma$ is a $(1, C_2)$-quasi-geodesic in $G/\exp(V_{s-1} \oplus V_s)$ with $C_2 = (r+2)C_1 = (r+2)^2 C$. We repeat until after $(s-1)$ steps we get that $\pi \circ \gamma = \pi_1 \circ \cdots \circ \pi_{s-1} \circ \gamma$ is a $(1, (r+2)^{s-1} C)$-quasigeodesic.

As with Theorem 3.2, the upper bound follows immediately from the projection π_{s-1} being 1-Lipschitz. Thus it is enough to show the lower bound $|b - a| - C_1 \leq d(\pi_{s-1} \circ \gamma(a), \pi_{s-1} \circ \gamma(b))$, for all $a, b \in \mathbb{R}$.

Set $\Gamma := \gamma(\mathbb{R} \setminus [a,b])$ and fix an arbitrary basepoint $\bar{t} \in \mathbb{R} \setminus [a,b]$. Since (4.2.i) does not hold for γ, the same is true for any translation of γ. Therefore we can assume without loss of generality that $\gamma(\bar{t}) = 1_G$.

Fix an arbitrary $\epsilon > 0$. Let $K > 0$ be the constant of Proposition 2.26 and let $c > 0$ be the constant of Lemma 2.32. Since $\gamma([a,b])$ is a bounded set, the failure of (4.2.i) for γ implies that Γ is also not contained in any neighborhood of any hyperplane. Since $G/[G,G]$ and $\mathbb{R}^r$ are biLipschitz equivalent, Lemma 2.31 implies that $\mathrm{MinHeight}(\pi(P))$ is not bounded as P varies in Γ^r. In particular, we may fix some $P \in \Gamma^r$ such that

$$\mathrm{MinHeight}(\pi(P)) > \frac{K^{s-1} d(\gamma(a),\gamma(b))^s}{c\epsilon^{s-1}}. \tag{4.3}$$

Consider the tuple $E := (\gamma(t_0), \dots, \gamma(t_{r+2}))$, where

$$\{\gamma(t_0), \dots, \gamma(t_{r+2})\} = P \cup \{\gamma(\bar{t}), \gamma(a), \gamma(b)\},$$

with the times t_j ordered so that $t_0 < \cdots < t_{r+2}$.

By the definition of Γ and $\bar{t}$, the points $\gamma(a)$ and $\gamma(b)$ are necessarily consecutive in this ordering, so there is some $\ell \in \{1, \dots, r+2\}$ such that $t_{\ell-1} = a$ and $t_\ell = b$. Denote by $E_P \in \Gamma^{r+1}$ the tuple E without $\gamma(a)$ and $\gamma(b)$, i.e.,

$$E_P := (\gamma(t_0), \dots, \gamma(t_{\ell-2}), \gamma(t_{\ell+1}), \dots, \gamma(t_{r+2})).$$

Applying Proposition 2.26 with the above E and ℓ, we get the bound

$$\begin{aligned} d(\gamma(t_0),\gamma(t_{r+2})) \leq d(\pi_{s-1}\circ\gamma(a), \pi_{s-1}\circ\gamma(b)) + \sum_{j\neq\ell} d(\gamma(t_{j-1}),\gamma(t_j)) \\ + K\left(\frac{d(\gamma(a),\gamma(b))^s}{\mathrm{Size}(E_P)}\right)^{\frac{1}{s-1}}. \end{aligned} \tag{4.4}$$

Estimating the distances along γ by (4.1) gives

$$\sum_{j\neq\ell} d(y_{j-1},y_j) \leq \sum_{j\neq\ell} |t_{j-1}-t_j| + (r+1)C = |t_0 - t_{r+2}| - |a-b| + (r+1)C$$

and

$$d(\gamma(t_0),\gamma(t_{r+2})) \geq |t_0 - t_{r+2}| - C.$$

Applying the above distance estimates to (4.4) and reorganizing terms, we get the lower bound

$$d(\pi_{s-1}\circ\gamma(a), \pi_{s-1}\circ\gamma(b)) \geq |a-b| - (r+2)C - K\left(\frac{d(\gamma(a),\gamma(b))^s}{\mathrm{Size}(E_P)}\right)^{\frac{1}{s-1}}, \tag{4.5}$$

which is exactly the desired lower bound except for the final term.

However, since $\gamma(\bar{t}) = 1_G$, applying Lemma 2.32 with ℓ such that $t_\ell = \bar{t}$ gives

$$\mathrm{Size}(E_P) \geq c \cdot \mathrm{MinHeight}(\pi(P)). \tag{4.6}$$

Bounding $\mathrm{Size}(E_P)$ by (4.6) and $\mathrm{MinHeight}(\pi(P))$ by (4.3), the lower bound (4.5) is simplified to

$$d(\pi_{s-1} \circ \gamma(a), \pi_{s-1} \circ \gamma(b)) \geq |a - b| - (r+2)C - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have the desired quasi-geodesic lower bound. q.e.d.

The second possible conclusion (4.2.ii) in Theorem 4.2 is that $\pi \circ \gamma$ is a quasi-geodesic in the normed space $G/[G,G]$. We next show that in the case of an inner product space, quasi-geodesics are well behaved on the large scale. Namely, every rough geodesic in $\mathbb{R}^n$ has a unique asymptotic cone and this asymptotic cone is a line. The result can also be deduced from [**RX02**, Theorem 2.5]. Note that not all $(1, C)$-quasi-geodesics are a finite distance from a line, since for example graphs of 1/2-Hölder functions $\mathbb{R} \to \mathbb{R}$ are $(1, C)$-quasi-geodesics in $\mathbb{R}^2$.

Lemma 4.7. *Every $(1, C)$-quasi-geodesic $\gamma : \mathbb{R} \to \mathbb{R}^n$ in a Euclidean space $\mathbb{R}^n$ has a unique blowdown and the blowdown is a line.*

Proof. Translating and reparametrizing if necessary, we may assume that $\gamma(0) = 0$. We denote by $\angle(t, s)$ the angle formed by $\gamma(t)$ and $\gamma(s)$ at 0. Its magnitude is given by the standard inner product on $\mathbb{R}^n$ via

$$\cos \angle(t, s) = \frac{\gamma(t) \cdot \gamma(s)}{|\gamma(t)|\,|\gamma(s)|}. \tag{4.8}$$

We first show that as $t, s \to \infty$, the angle vanishes, i.e., $1 - \cos \angle(t, s) \to 0$. By symmetry we can assume that $t \geq s \geq 0$.

In an inner product space we have for all x, y the identity

$$2\,|x|\,|y| - 2x \cdot y = |x - y|^2 - (|x| - |y|)^2.$$

Combining (4.8) and the above identity for $x = \gamma(t)$ and $y = \gamma(s)$, we get

$$\begin{aligned} 1 - \cos \angle(t, s) &= \frac{2\,|\gamma(t)|\,|\gamma(s)| - 2\gamma(t) \cdot \gamma(s)}{2\,|\gamma(t)|\,|\gamma(s)|} \\ &= \frac{|\gamma(t) - \gamma(s)|^2 - (|\gamma(t)| - |\gamma(s)|)^2}{2\,|\gamma(t)|\,|\gamma(s)|}. \end{aligned} \tag{4.9}$$

The quasi-geodesic bound (4.1) and the assumption $t \geq s \geq 0$ imply that

$$\begin{aligned} |\gamma(t) - \gamma(s)|^2 - (|\gamma(t)| - |\gamma(s)|)^2 &\leq (t - s + C)^2 - (t - s - 2C)^2 \\ &= 6C(t - s) - 3C^2 \leq 6Ct. \end{aligned}$$

Moreover, the bound (4.1) implies also that when $t, s \geq 2C$ we have

$$|\gamma(t)|\,|\gamma(s)| \geq (t-C)(s-C) \geq \frac{1}{4}ts.$$

Estimating (4.9) using the above two inequalities, we get for all $t, s \geq 2C$ the upper bound

$$1 - \cos\angle(t,s) \leq \frac{6t}{\frac{1}{4}ts} = \frac{24}{s}$$

and hence $\angle(t,s) \to 0$ as $t \geq s \to \infty$. Repeating a similar argument for $t \leq s \leq 0$, we see also that $\angle(t,s) \to 0$ as $t \leq s \to -\infty$.

From this estimate of angles we conclude that the limit directions $v_+ = \lim\limits_{t\to\infty} \gamma(t)/\,|\gamma(t)|$ and $v_- = \lim\limits_{t\to-\infty} \gamma(t)/\,|\gamma(t)|$ always exist. We claim that this implies that the asymptotic cone $\lim\limits_{h\to\infty} \gamma_h$ exists without taking any subsequences, thus proving uniqueness.

First, observe that the existence of the limit direction and γ being a quasi-geodesic implies that also $\lim\limits_{t\to\infty} \gamma(t)/t = v_+$. Indeed, for any $t > C$, by (4.1), we have

$$\left|\frac{\gamma(t)}{t} - \frac{\gamma(t)}{|\gamma(t)|}\right| = \frac{|\gamma(t)|\,\big||\gamma(t)| - t\big|}{t\,|\gamma(t)|} \leq \frac{(t+C)C}{t(t-C)} \to 0$$

as $t \to \infty$. This implies that $\lim\limits_{h\to\infty} \gamma_h(1) = v_+$. For arbitrary $t > 0$,

$$\lim_{h\to\infty} \gamma_h(t) = \lim_{h\to\infty} \frac{\gamma(ht)}{h} = t \lim_{h\to\infty} \frac{\gamma(ht)}{ht} = tv_+.$$

Similarly $\lim\limits_{h\to\infty} \gamma_h(t) = -tv_-$ for all $t < 0$, proving existence and uniqueness of the blowdown.

To see that the unique limit is a line, i.e., that $v_- = -v_+$, it suffices to observe that any blowdown of a $(1,C)$-quasi-geodesic in $\mathbb{R}^n$ is a geodesic in $\mathbb{R}^n$, and geodesics in $\mathbb{R}^n$ are lines. q.e.d.

Combining Theorem 4.2 with Lemma 4.7 allows us to conclude the lower rank subgroup containment for blowdowns of rough geodesics in sub-Riemannian Carnot groups:

Corollary 4.10. *If $\gamma : \mathbb{R} \to G$ is a $(1,C)$-quasi-geodesic in a sub-Riemannian Carnot group $G \neq \mathbb{R}$, then there exists a proper Carnot subgroup $H < G$ containing every element of* $\mathrm{Asymp}(\gamma)$.

Proof. Consider the two cases of Theorem 4.2. In the first case (4.2.i), the horizontal projection is in a finite neighborhood of a hyperplane, $\mathrm{Im}(\pi \circ \gamma) \subset B_{V_1}(W,R)$. Thus any blowdown $\sigma \in \mathrm{Asymp}(\gamma)$ has its horizontal projection completely contained in W. Since $\sigma(0) = 1_G$, it follows that σ is contained in the Carnot subgroup H generated by W. The rank of H is by construction the dimension of W, which is smaller than the rank of G.

In the second case (4.2.ii), the horizontal projection $\pi \circ \gamma$ is a $(1, C')$-quasi-geodesic. Thus by Lemma 4.7 it has a unique blowdown σ, which is a line. But then $H := \sigma(\mathbb{R})$ is itself a one-parameter subgroup containing all blowdowns, proving the claim. q.e.d.

5. Infinite geodesics

5.1. Abnormality of blowdowns of geodesics. Let G be a sub-Riemannian Carnot group, so that on the first layer V_1 of the Lie algebra $\mathfrak{g}$ we have an inner product $\langle \cdot, \cdot \rangle$. Every geodesic $\gamma : I \to G$ on a finite interval $I \subset \mathbb{R}$ is then a solution of the Pontryagin maximum principle. In sub-Riemannian Carnot groups the principle takes the form

$$\text{(PMP)} \qquad \lambda \left(\int_I \mathrm{Ad}_{\gamma(t)}\, v(t)\, dt \right) = \xi \, \langle u_\gamma, v \rangle \quad \forall v \in L^2(I; V_1),$$

for some $\lambda \in \mathfrak{g}^*$ and $\xi \in \mathbb{R}$ such that $(\lambda, \xi) \neq (0,0)$, see **[LMO$^+$16]** for the calculation of the differential of the endpoint map. Here, $u_\gamma \in L^2(I; V_1)$ denotes the control of γ.

A curve is abnormal exactly when it satisfies PMP with $\xi = 0$ for some $\lambda \in \mathfrak{g}^* \setminus \{0\}$. In the case of a geodesic $\gamma : J \to \mathbb{R}$ on an unbounded interval $J \subset \mathbb{R}$, there exists a pair $(\lambda, \xi) \neq (0,0)$ for which PMP is satisfied for every bounded subinterval $I \subset J$.

In this section we will consider properties of asymptotic cones of geodesics from the point of view of the Pontryagin maximum principle. The next lemma describes what happens to PMP for dilations of geodesics.

Lemma 5.1. *Let $\gamma : I \to G$ be a horizontal curve in G that satisfies PMP for a pair (λ, ξ). Then for any $h > 0$, the dilated curve $\gamma_h : I_h \to G$ satisfies PMP for the pair $(\delta_h^* \lambda, h\xi)$.*

Proof. We may suppose without loss of generality that the interval I is bounded and the dilation

$$\gamma_h(t) = \delta_{1/h} \circ \gamma(\bar{t} + ht)$$

is happening at $\bar{t} = 0$. The dilations are homomorphisms, so by the definition of Ad_g as the differential of the conjugation $x \mapsto gxg^{-1}$, the map $\mathrm{Ad}_{\gamma(t)}$ can be written in terms of $\mathrm{Ad}_{\gamma_h(t)}$ as

$$\mathrm{Ad}_{\gamma(t)} = \mathrm{Ad}_{\delta_h \circ \gamma_h(t/h)} = (\delta_h)_* \circ \mathrm{Ad}_{\gamma_h(t/h)} \circ (\delta_{1/h})_*.$$

Therefore, PMP for γ gives the identity

$$\xi \, \langle u_\gamma, v \rangle = \lambda \left(\int_I \mathrm{Ad}_{\gamma(t)}\, v(t)\, dt \right) = (\delta_h^* \lambda) \left(\int_I \mathrm{Ad}_{\gamma_h(t/h)} \frac{1}{h} v(t)\, dt \right). \tag{5.2}$$

Denote for each $v \in L^2(I; V_1)$ by $\tilde{v} \in L^2(I_h; V_1)$ the reparametrized function $\tilde{v}(t) = v(ht)$. Then after a change of variables, the right hand side of (5.2) is

$$\int_I \mathrm{Ad}_{\gamma_h(t/h)} \frac{1}{h} v(t)\, dt = \int_{I_h} \mathrm{Ad}_{\gamma_h(t)} \tilde{v}(t)\, dt. \tag{5.3}$$

Since the control u_h of the dilated curve γ_h is

$$u_h(t) = (\delta_{1/h})_* u_\gamma(ht) \cdot h = u_\gamma(ht),$$

a similar change of variables as in (5.3) shows that

$$\begin{aligned} \langle u_\gamma, v\rangle &= \int_I u_\gamma(t) v(t)\, dt = \int_I u_h(t/h)\tilde{v}(t/h)\, dt \\ &= h \int_{I_h} u_h(t)\tilde{v}(t)\, dt = h\, \langle u_h, \tilde{v}\rangle\, . \end{aligned} \tag{5.4}$$

Applying both changes of variables (5.3) and (5.4) to (5.2) gives the identity

$$h\xi\, \langle u_h, \tilde{v}\rangle = (\delta_h^* \lambda) \left(\int_{I_h} \mathrm{Ad}_{\gamma_h(t)} \tilde{v}(t)\, dt \right).$$

Since every element of $L^2(I_h; V_1)$ can be written as $\tilde{v}$ for some $v \in L^2(I; V_1)$, the above shows that γ_h satifies PMP for the pair $(\delta_h^* \lambda, h\xi)$. q.e.d.

In every sub-Finsler Carnot group horizontal lines through the identity are infinite geodesics that are dilation invariant. Hence, the unique blowdown of any horizontal line is the line itself translated to the identity, which may or may not be abnormal. For all other curves however, every blowdown is necessarily an abnormal curve:

Proposition 5.5. *In sub-Riemannian Carnot groups asymptotic cones of non-line infinite geodesics are abnormal curves.*

Proof. The argument is partially inspired by [**Agr98**]. Let γ be a geodesic in G and let $(\lambda, \xi) \in \mathfrak{g}^* \times \mathbb{R}$ be a pair for which γ satisfies PMP. We decompose λ as $\lambda = \lambda^{(1)} + \cdots + \lambda^{(s)} \in V_1^* \oplus \cdots \oplus V_s^* \simeq \mathfrak{g}^*$ and let $j \in \{1, \ldots, s\}$ be the largest index for which $\lambda^{(j)} \neq 0$.

If $\lambda^{(2)} = \cdots = \lambda^{(s)} = 0$, then PMP reduces to

$$\lambda\Big(\int_I v \Big) = \xi\, \langle u_\gamma, v\rangle \quad \forall v \in L^2(I; V_1)$$

on every finite interval $I \subset \mathbb{R}$. Thus if $\lambda^{(2)} = \cdots = \lambda^{(s)} = 0$, then u_γ is constant and γ is a line. Assume from now on that γ is not a line, so $j \geq 2$.

By Lemma 5.1 the dilated curve γ_h satisfies PMP for the pair $(\delta_h^* \lambda, h\xi)$. In terms of the decomposition into layers, we have

$$\delta_h^* \lambda = \delta_h^* (\lambda^{(1)} + \cdots + \lambda^{(j)}) = h\lambda^{(1)} + \cdots + h^j \lambda^{(j)}.$$

Note that PMP is scale invariant with respect to the covector pair. Therefore scaling by $\frac{1}{h^j}$, we see that γ_h satisfies PMP also for the pair

$\frac{1}{h^j}(\delta_h^*\lambda, h\xi) = (\frac{1}{h^j}\delta_h^*\lambda, \frac{1}{h^{j-1}}\xi)$. These pairs form a convergent sequence as $h \to \infty$:

$$\lim_{h\to\infty}(\frac{1}{h^j}\delta_h^*\lambda, \frac{1}{h^{j-1}}\xi) = (\lambda_\infty, 0),$$

where

$$\lambda_\infty := \lim_{h\to\infty}(h^{1-j}\lambda^{(1)} + h^{2-j}\lambda^{(2)} + \cdots + \lambda^{(j)}) = \lambda^{(j)} \neq 0.$$

Let $\sigma \in \operatorname{Asymp}(\gamma)$, so there exists some sequence $h_j \to \infty$ for which $\sigma = \lim_{j\to\infty}\gamma_{h_j}$. By continuity, it follows that σ satisfies PMP for the pair $(\lambda_\infty, 0)$, so σ is an abnormal curve. q.e.d.

5.2. Rigidity of geodesics. Proposition 5.5 leads to the question of what abnormal infinite geodesics exist. Any curve in a Carnot group G is abnormal in a product $G\times H$, where H is any non-abelian Carnot group, so any infinite geodesic could be artificially made abnormal. Examples in non-product spaces can also be found by lifting such abnormals. However we are not aware of any examples of infinite abnormal geodesics in rank 2 Carnot groups, or even of *strictly* abnormal infinite geodesics in Carnot groups of any rank.

One issue in trying to find infinite (strictly) abnormal geodesics is that it is difficult to prove that any extremal is minimizing for all time. Currently known examples of non-line infinite geodesics such as the one studied in Section 7.2 arise from a complete optimal synthesis. This is however often unfeasible, as the Hamiltonian system for the normal equations is often non-integrable in higher dimensional problems, see for instance the non-integrability results in [**BBKM16**, **LS18**].

In sub-Riemannian Carnot groups of step 2 the only infinite geodesics are lines [**Kis03**, Proposition 2.2]. The same rigidity result can be extended to all sub-Finsler Carnot groups with a strictly convex norm [**Hak20**]. That is, if a strictly abnormal infinite geodesic exists, it must be found in step 3 or above.

6. Applications of the existence of a line tangent

We next provide some consequences of the existence of a line tangent.

6.1. Loss of optimality. We prove that there are geodesics that lose optimality whenever they are extended.

Proposition 6.1. *In every non-Abelian sub-Finsler Carnot group defined by a strictly convex norm (e.g., in every sub-Riemannian Carnot group) there exist finite-length geodesics that cannot be extended as geodesics.*

Proof. For every such group G, we know that the only infinite geodesics in $G/[G,G]$ are lines. Therefore, by Corollary 1.4 every geodesic has an iterated tangent that is a line. Since iterated tangents are tangents, we have that every geodesic in G has a line tangent.

Fix a nonzero element $v \in V_2$, which exists since G is not Abelian. Let $\gamma : [0,T] \to G$ be a geodesic with $\gamma(0) = 1_G$ and $\gamma(T) = \exp(v)$. We claim that any such geodesic cannot be extended to a geodesic $\tilde{\gamma} : [-\epsilon, T] \to G$ such that $\tilde{\gamma}|_{[0,T]} = \gamma$ for any $\epsilon > 0$.

Let $\delta_{-1} : G \to G$ be the group homomorphism such that $(\delta_{-1})_*(v) = (-1)^j v$ for all $v \in V_j$. The map δ_{-1} is an isometry, since $(\delta_{-1})_*|_{V_1}$ is an isometry. Notice that $\delta_{-1} \circ \gamma$ is another[2] geodesic from 1_G to $\exp(v)$.

Suppose that an extension $\tilde{\gamma} : [\epsilon, T] \to G$ of γ existed. By the existence of a line tangent outlined in the first paragraph, we have that there exists a sequence $h_j \to 0$ such that

$$\tilde{\gamma}_{h_j} = \delta_{\frac{1}{h_j}} \circ \tilde{\gamma} \circ \delta_{h_j} \to \sigma,$$

with $\sigma(t) = \exp(tX)$ for some $X \in V_1$. Replace γ by $\delta_{-1} \circ \gamma$ in the extension $\tilde{\gamma}$, i.e., consider the concatenated curve

$$\eta := \tilde{\gamma}|_{[-\epsilon,0]} * (\delta_{-1} \circ \gamma).$$

Since γ and $\delta_{-1} \circ \gamma$ are both geodesics with the same endpoints, and $\tilde{\gamma}$ was a geodesic extension of γ, the curve η is also a geodesic. However, η has a blowup at 0 that is not injective: for $t < 0$,

$$\eta_{\epsilon_j}(t) = (\delta_{\frac{1}{\epsilon_j}} \circ \eta \circ \delta_{\epsilon_j})(t) = \tilde{\gamma}_{\epsilon_j}(t) \to \exp(tX)$$

whereas for $t > 0$,

$$\eta_{\epsilon_j}(t) = (\delta_{\frac{1}{\epsilon_j}} \circ \delta_{-1} \circ \gamma \circ \delta_{\epsilon_j})(t) = \delta_{-1} \circ \tilde{\gamma}_{\epsilon_j}(t) \to \delta_{-1} \exp(tX) = \exp(-tX).$$

Any blowup of the geodesic η would have to be a geodesic, but this blowup is not even injective, so we get a contradiction. q.e.d.

6.2. Non-minimality of corners for distributions of non-constant rank. In the previous work [**HL16**] we proved that corners cannot be length-minimizing in any sub-Riemannian manifold, in which by standard definition the distribution has constant rank. We will next show that the existence of a line tangent implies that the assumption that the distribution has constant rank can be omitted.

Proposition 6.2. *Let M be a generalized sub-Riemannian manifold with a distribution not necessarily of constant rank. If $\gamma : I \to M$ is a geodesic, then γ cannot have a corner-type singularity.*

Proof. Reparametrizing and translating, it suffices to consider the case when $I = (-\epsilon, \epsilon)$ and show that γ cannot have a corner-type singularity at 0.

Let $\tilde{M}$ be a desingularization of M as in [**Jea14**, Lemma 2.5], that is an equiregular sub-Riemannian manifold equipped with a canonical

[2] We learned this trick for proving non-uniqueness of geodesics in Carnot groups from [**Ber16**, Proposition 3.2]

projection $\pi : \tilde{M} \to M$. Since the projection is a submetry, the geodesic γ can be lifted to a geodesic $\tilde{\gamma} : I \to \tilde{M}$. Let $u : (-\epsilon, \epsilon) \to \mathbb{R}^r$ be the control of $\tilde{\gamma}$ with respect to a fixed frame $\tilde{X}_1, \dots, \tilde{X}_r$ of horizontal vector fields on $\tilde{M}$. Then u is also the control of γ with respect to the projected horizontal frame $\pi_* \tilde{X}_1, \dots, \pi_* \tilde{X}_r$ on M.

By Theorem 1.1, the curve $\tilde{\gamma}$ has an iterated tangent that is a line, and thus also a tangent that is a line. By [**MPV18b**, Remark 3.12] it follows that there exist a constant $v \in \mathbb{R}^r$ and a sequence of scales $h_j \to 0$ such that for the rescaled controls $u^{(j)} : (-\epsilon/h_j, \epsilon/h_j) \to \mathbb{R}^r$, $u^{(j)}(t) = u(h_j t)$, we have $u^{(j)} \to v$ in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^r)$.

On the other hand, in coordinates near $\gamma(0)$ on M, for any small enough h_j we have

$$\frac{\gamma(h_j)}{h_j} = \int_0^{h_j} \sum_{k=1}^r u_k(t) X_k(\gamma(t)) \, \frac{dt}{h_j} = \int_0^1 \sum_{k=1}^r u_k^{(j)}(t) X_k(\gamma(h_j t)) \, dt \quad \text{and}$$

$$\frac{\gamma(-h_j)}{-h_j} = \int_{-h_j}^0 \sum_{k=1}^r u_k(t) X_k(\gamma(t)) \, \frac{dt}{h_j} = \int_{-1}^0 \sum_{k=1}^r u_k^{(j)}(t) X_k(\gamma(h_j t)) \, dt.$$

By continuity of the vector fields $X_1, \dots, X_r$ and the convergence $u^{(j)} \to v$ in L^2_{loc}, we see that

$$\lim_{j \to \infty} \frac{\gamma(h_j)}{h_j} = \sum_{k=1}^r v_k X_k(\gamma(0)) = \lim_{j \to \infty} \frac{\gamma(-h_j)}{-h_j}.$$

In particular, if γ has one-sided derivatives at 0, then they must be equal, so γ cannot have a corner-type singularity. q.e.d.

7. On sharpness of Theorem 1.5

In this section we want to consider whether Theorem 1.5 can be improved. In particular, we will show that in the statement of the theorem, taking the horizontal projection is essential. That is, there exist geodesics that are not in a finite neighborhood of any proper Carnot subgroup (see Corollary 7.28).

A possible improvement of Theorem 1.5 would be to strengthen the claim in the horizontal projection. Namely, the following might be true.

Conjecture 7.1. *If $\gamma : \mathbb{R} \to G$ is a geodesic such that $\pi \circ \gamma : \mathbb{R} \to G/[G,G]$ is not a geodesic, then there exists a hyperplane $W \subset V_1$ such that $\lim_{t \to \pm\infty} d(\pi \circ \gamma(t), W) = 0$.*

Toward this conjecture, we shall consider the case of rank 2 Carnot groups, where proper Carnot subgroups are simply lines. For this reason, in the next subsection we first prove some general statements about lines that are a finite distance apart.

7.1. Lines in Carnot groups. A *line* in a Lie group is a left-translation of a one-parameter subgroup, i.e., a curve $L : \mathbb{R} \to G$ such that $L(t) = g \exp(tX)$ for some $g \in G$ and $X \in \mathfrak{g}$. We stress that in case G is a Carnot group the vector X is not assumed to be horizontal.

The distance between lines will be measured by the Hausdorff distance: The Hausdorff distance of two subsets $A, B \subset G$ is

$$d_H(A, B) := \max\Big(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big).$$

In Lemma 7.5 we will give two equivalent algebraic conditions for two lines to be at a bounded distance from each other. In the proof we will want to use also the notion of distance of lines given by the sup-norm, which is parametrization dependent. For this reason we first prove a sufficient condition (Lemma 7.2) for the equivalence of boundedness of Hausdorff distance and boundedness of sup-norm. This result is naturally stated in much more generality than just lines in Carnot groups.

Lemma 7.2. *Let X and Y be metric spaces, and let $\alpha : X \to Y$ and $\beta : X \to Y$ be maps such that the following conditions hold.*

(a) The map β is bornologous: For every $R < \infty$, there exists $R' < \infty$ such that $\beta(B_X(x, R)) \subset B_Y(\beta(x), R')$ for any $x \in X$.
(b) The map $\alpha \times \beta : X^2 \to Y^2$ maps distant points to distant points: For every $M < \infty$, there exists $R < \infty$ such that $d(\alpha(x_1), \beta(x_2)) > M$ for any x_1, x_2 with $d_X(x_1, x_2) > R$.

Then $d_H(\alpha(X), \beta(X)) < \infty$ if and only if $\sup_{x\in X} d(\alpha(x), \beta(x)) < \infty$.

Proof. Clearly $d_H(\alpha(X), \beta(X)) \le \sup_{x\in X} d(\alpha(x), \beta(x))$ so it suffices to prove the "only if" implication. That is, we assume that $M := d_H(\alpha(X), \beta(X)) < \infty$ and we will show that also $\sup_{x\in X} d(\alpha(x), \beta(x)) < \infty$.

By the definition of the Hausdorff distance, we have $d(\alpha(x), \beta(X)) \le M$ for every $x \in X$. Therefore there exists a (possibly discontinuous) map $f : X \to X$ choosing roughly closest points from $\beta(X)$, i.e., a map such that

$$d(\alpha(x), \beta \circ f(x)) \le M + 1 \quad \forall x \in X. \tag{7.3}$$

Let $R < \infty$ be the constant given by the assumption (b) such that $d(\alpha(x_1), \beta(x_2)) > M + 1$ for any $x_1, x_2 \in X$ with $d(x_1, x_2) > R$. Then the bound (7.3) implies that

$$d(x, f(x)) \le R \quad \forall x \in X.$$

Assumption (a) then implies that there exists $R' < \infty$ such that

$$d(\beta(x), \beta \circ f(x)) \le R' \quad \forall x \in X. \tag{7.4}$$

Combining the bounds (7.3) and (7.4), we get for any $x \in X$ the uniform bound

$$d(\alpha(x), \beta(x)) \leq d(\alpha(x), \beta \circ f(x)) + d(\beta \circ f(x), \beta(x)) \leq M + 1 + R' < \infty,$$

proving the claim. q.e.d.

Lemma 7.5. *Assume G is a Carnot group and let $L_1(t) = g\exp(tX)$ and $L_2(t) = h\exp(tY)$ be two lines in the group. The following are equivalent:*

(i) There exists a constant $c > 0$ such that $X = c\operatorname{Ad}_{g^{-1}h} Y$.
(ii) There exist a constant $c > 0$ and an element $k \in G$ such that $L_1(t) = L_2(ct)k$.
(iii) $d_H(L_1(\mathbb{R}_+), L_2(\mathbb{R}_+)) < \infty$.

Proof. The equivalence of (i) and (ii) is an algebraic computation: For any $k \in G$ and $Z \in \mathfrak{g}$, we have the identity

$$k\exp(Z) = k\exp(Z)k^{-1}k = C_k(\exp(Z)) \cdot k = \exp(\operatorname{Ad}_k Z)k.$$

For any $c > 0$, we apply the above with $k = g^{-1}h$ and $Z = ctY$. This gives the identity

$$\begin{aligned} L_1(t)^{-1}L_2(ct) &= (g\exp(tX))^{-1} \cdot (h\exp(ctY)) \\ &= \exp(-tX)\exp(ct\operatorname{Ad}_{g^{-1}h} Y)g^{-1}h. \end{aligned} \tag{7.6}$$

If (i) holds, then (7.6) implies that $L_1(t)^{-1}L_2(ct)$ is constant, proving (ii). Vice versa, if (ii) holds, then $L_1(t)^{-1}L_2(ct)$ is constant, so (7.6) is constant. But this is only possible if (i) holds.

That (ii) implies (iii) is immediate from the left-invariance of the distance on G. It remains to prove that (iii) implies (ii). The claim is equivalent to saying that the product $L_1(t)^{-1}L_2(ct)$ is constant for some $c > 0$. Since the product is in exponential coordinates a polynomial expression, it suffices to show that

$$\sup_{t \in \mathbb{R}_+} d(L_1(t), L_2(ct)) < \infty. \tag{7.7}$$

We will prove this by induction on the step of the group G. In a normed space, two half-lines are a finite distance apart if and only if they are parallel, so the claim holds in step 1.

Suppose that the claim is true for all Carnot groups of step at most $s - 1$ and suppose that G is of step s. We will prove (7.7) by applying Lemma 7.2 to the curves $\alpha(t) = L_2(ct)$ and $\beta(t) = L_1(t)$ for some $c > 0$ to be fixed later.

From the identity

$$d(L_1(t_2), L_1(t_1)) = d(L_1(0), L_1(t_1 - t_2))$$

we see that $R' = \sup_{|t| \leq R} d(L_1(0), L_1(t)) < \infty$ satisfies assumption (a) of Lemma 7.2.

For assumption (b) of Lemma 7.2, we need a lower bound for the value $d(L_1(t_1), L_2(ct_2))$. We consider first the case when the lines degenerate under the projection $\pi_{s-1} : G \to G/\exp(V_s)$ to step $s-1$, i.e., when $X, Y \in V_s$. Since elements in $\exp(V_s)$ commute with everything, for any $t_1, t_2 \in \mathbb{R}_+$ we have that

$$d(L_1(t_1), L_2(t_2)) = d(1_G, g^{-1}h\exp(t_2Y - t_1X)).$$

If $Y = cX$ for some $c > 0$, then condition (i) is satisfied, which implies (7.7) by the first part of the proof. Otherwise, $t_2Y - t_1X$ escapes any compact subset of V_s as $|t_2 - t_1| \to \infty$. Recall that in Carnot groups the exponential map is a global diffeomorphism and the distance function is proper. Hence, the lower bound

$$d(L_1(t_1), L_2(t_2)) \geq d(1_G, \exp(t_2Y_2 - t_1Y_1)) - d(1_G, g^{-1}h)$$

implies that assumption (b) of Lemma 7.2 is satisfied for any $c > 0$. By Lemma 7.2 we conclude that in this case (7.7) is satisfied for any $c > 0$.

Next we consider the case when at least one of the lines does not degenerate under the projection $\pi_{s-1} : G \to G/\exp(V_s)$. Since the projection π_{s-1} is 1-Lipschitz, we have

$$d_H(\pi_{s-1} \circ L_1(\mathbb{R}_+), \pi_{s-1} \circ L_2(\mathbb{R}_+)) \leq d_H(L_1(\mathbb{R}_+), L_2(\mathbb{R}_+)) < \infty.$$

Note that the above implies that also the other line cannot degenerate to a constant.

By the inductive assumption in the step $s-1$ Carnot group $G/\exp(V_s)$, we can fix $c > 0$ such that

$$M := \sup_{t \in \mathbb{R}_+} d(\pi_{s-1} \circ L_1(t), \pi_{s-1} \circ L_2(ct)) < \infty.$$

It follows that for any $t_1, t_2 \in \mathbb{R}_+$ we get the lower bound

$$\begin{aligned} d(L_1(t_1), L_2(ct_2)) &\geq d(\pi_{s-1} \circ L_1(t_1), \pi_{s-1} \circ L_2(ct_2)) \\ &\geq d(\pi_{s-1} \circ L_1(t_1), \pi_{s-1} \circ L_1(t_2)) \\ &\quad - d(\pi_{s-1} \circ L_1(t_2), \pi_{s-1} \circ L_2(ct_2)) \\ &\geq d(\pi_{s-1} \circ L_1(t_1), \pi_{s-1} \circ L_1(t_2)) - M. \end{aligned} \tag{7.8}$$

Decompose the direction vector of L_1 into homogeneous components as $X = X_{(1)} + \cdots + X_{(s)} \in V_1 \oplus \cdots \oplus V_s$ and let k be the smallest index for which $X_{(k)} \neq 0$. Since $\pi_{s-1} \circ L_1$ is non-constant, we have $k \leq s-1$. By homogeneity of the distance in the projection to step k we get the lower bound

$$\begin{aligned} d(\pi_{s-1} \circ L_1(t_1), \pi_{s-1} \circ L_1(t_2)) &\geq d(\pi_k \circ L_1(0), \pi_k \circ L_1(t_2 - t_1)) \\ &= d(\pi_k(1_G), \pi_k \circ \exp((t_2 - t_1)X_{(k)})) \\ &= |t_2 - t_1|^{1/k}\, d(\pi_k(1_G), \pi_k \circ \exp(X_{(k)})). \end{aligned} \tag{7.9}$$

Combining (7.8) and (7.9) and denoting $C := d(\pi_k(1_G), \pi_k \circ \exp(X_{(k)})) > 0$, we have that

$$d(L_1(t_1), L_2(ct_2)) \geq C\, |t_2 - t_1|^{1/k} - M.$$

This shows that assumption (b) of Lemma 7.2 holds for $\alpha(t) = L_2(ct)$ and $\beta(t) = L_1(t)$, so Lemma 7.2 implies that we have (7.7). q.e.d.

7.2. An explicit infinite non-line geodesic in the Engel group. The sub-Riemannian Engel group E is a sub-Riemannian Carnot group of rank 2 and step 3 of dimension 4. Its Lie algebra $\mathfrak{g}$ has a basis $X_1, X_2, X_{12}, X_{112}$ whose only non-zero commutators are

$$[X_1, X_2] = X_{12} \quad \text{and} \quad [X_1, X_{12}] = X_{112}.$$

In [**AS15**], Ardentov and Sachkov studied the cut time for normal extremals in the Engel group, and found a family of infinite geodesics that are not lines. These geodesics have a property stronger than that implied by Theorem 1.5. Namely, instead of merely having their horizontal projections contained in a finite neighborhood of a hyperplane, their horizontal projections are in fact asymptotic to a line.

To study these infinite geodesics explicitly, we will consider exponential coordinates $\mathbb{R}^4 \to E$ defined by

$$x = (x_1, x_2, x_{12}, x_{112}) \mapsto \exp(x_1X_1 + x_2X_2 + x_{12}X_{12} + x_{112}X_{112}).$$

By the BCH formula, the group law is given by $x \cdot y = z$, where

$$\begin{aligned} z_1 &= x_1 + y_1 \\ z_2 &= x_2 + y_2 \\ z_{12} &= x_{12} + y_{12} + \frac{1}{2}(x_1y_2 - x_2y_1) \\ z_{112} &= x_{112} + y_{112} + \frac{1}{2}(x_1y_{12} - x_{12}y_1) \\ &\quad + \frac{1}{12}(x_1^2y_2 - x_1x_2y_1 - x_1y_1y_2 + x_2y_1^2). \end{aligned} \tag{7.10}$$

The left-invariant extensions of the horizontal basis vectors X_1, X_2 are

$$\begin{aligned} X_1(x) &= \partial_1 - \frac{1}{2}x_2\partial_{12} - (\frac{1}{12}x_1x_2 + \frac{1}{2}x_{12})\partial_{112} \quad \text{and} \\ X_2(x) &= \partial_2 + \frac{1}{2}x_1\partial_{12} + \frac{1}{12}x_1^2\partial_{112}. \end{aligned} \tag{7.11}$$

Note that the coordinates used in [**AS15**] are not exponential coordinates, but the two coordinate systems agree in the horizontal (x_1 and x_2) components.

Given a covector written in the dual basis as $\lambda = (\lambda_1, \lambda_2, \lambda_{12}, \lambda_{112}) \in \mathfrak{g}^*$, the normal equation given by PMP takes the form

$$u_\gamma(t) = \lambda\left(\mathrm{Ad}_{\gamma(t)} X_1\right) X_1 + \lambda\left(\mathrm{Ad}_{\gamma(t)} X_2\right) X_2. \tag{7.12}$$

In [**AS15**], the space of covectors $\mathfrak{g}^*$ is stratified into 7 different classes $C_1, \ldots, C_7$ based on the different types of trajectories of the corresponding normal extremals. For our purposes the relevant class is C_3, which consists of the non-line infinite geodesics. In [**AS15**], the class was parametrized by

$$C_3 = \{(\cos(\theta+\pi/2), \sin(\theta+\pi/2), c, \alpha) : \ \alpha \neq 0, \frac{c^2}{2} - \alpha \cos\theta = |\alpha|, c \neq 0\}.$$

An example of a covector $\lambda \in \mathfrak{g}^*$ in the class C_3 is $\lambda = (0, 1, 2, 1)$. However, instead of integrating the normal equation (7.12) with this covector, we will consider a translation of the curve to simplify the asymptotic study of the resulting curve. Instead of considering the geodesic starting from $(0, 0, 0, 0)$, we will consider the translated geodesic starting from $(2, 0, 0, 0)$.

If $\gamma : \mathbb{R} \to E$ satisfies (7.12) with the covector λ, then a left-translation $\beta = g\gamma : \mathbb{R} \to E$ by $g \in E$ satisfies

$$\begin{aligned} u_\beta(t) &= \lambda\left(\operatorname{Ad}_{\gamma(t)} X_1\right) X_1 + \lambda\left(\operatorname{Ad}_{\gamma(t)} X_2\right) X_2 \\ &= \lambda\left(\operatorname{Ad}_{g^{-1}\beta(t)} X_1\right) X_1 + \lambda\left(\operatorname{Ad}_{g^{-1}\beta(t)} X_2\right) X_2. \end{aligned} \tag{7.13}$$

Using the formula

$$\operatorname{Ad}_{\exp(Y)} X = e^{\operatorname{ad}(Y)} X,$$

we compute for $x = (x_1, x_2, x_{12}, x_{112}) \in E$ that

$$\begin{aligned} \operatorname{Ad}_x X_1 &= X_1 - x_2 X_{12} - (x_{12} + \frac{1}{2} x_1 x_2) X_{112} \quad \text{and} \\ \operatorname{Ad}_x X_2 &= X_2 + x_1 X_{12} + \frac{1}{2} x_1^2 X_{112}. \end{aligned}$$

Evaluated for the covector $\lambda = (0, 1, 2, 1)$, we get

$$\begin{aligned} \lambda(\operatorname{Ad}_x X_1) &= -2x_2 - x_{12} - \frac{1}{2} x_1 x_2, \\ \lambda(\operatorname{Ad}_x X_2) &= 1 + 2x_1 + \frac{1}{2} x_1^2. \end{aligned} \tag{7.14}$$

By the group law (7.10), the translation of the curve in which we are interested is

$$(2, 0, 0, 0)^{-1} \cdot \beta = \left(\beta_1 - 2,\ \beta_2,\ \beta_{12} - \beta_2,\ \beta_{112} - \beta_{12} + \frac{1}{3}\beta_2 + \frac{1}{6}\beta_1\beta_2\right).$$

Substituting the points $x = (2, 0, 0, 0)^{-1} \cdot \beta(t)$ into (7.13) using (7.14), we get the ODE

$$\dot\beta_1 = -\frac{1}{2}\beta_1\beta_2 - \beta_{12} \qquad\qquad \dot\beta_2 = \frac{1}{2}\beta_1^2 - 1. \tag{7.15}$$

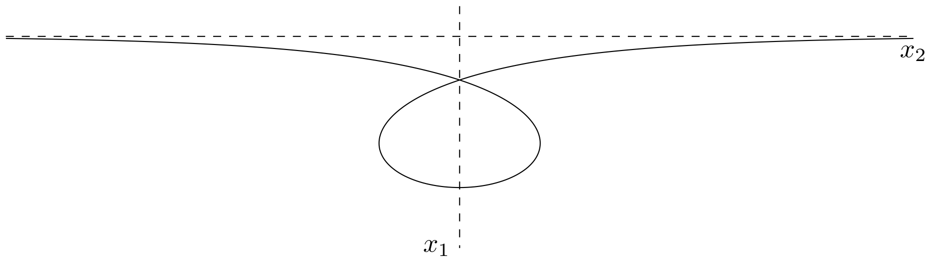

Figure 1. Horizontal projection of the non-line Engel infinite geodesic β (rotated 90° clockwise)

Lemma 7.16. *The horizontal curve $\beta : \mathbb{R} \to E$ satisfying the ODE* (7.15) *with the initial condition* $\beta(0) = (2, 0, 0, 0)$ *has the explicit form (see Figure 1)*

$$\beta_1(t) = \frac{2}{\cosh(t)}, \qquad \beta_2(t) = 2\tanh(t) - t,$$
$$\beta_{12}(t) = \frac{t}{\cosh(t)}, \qquad \beta_{112}(t) = \frac{2}{3}\tanh(t) - \frac{t}{3\cosh(t)^2}.$$

Proof. The proof of the lemma is a direct computation. First we shall verify horizontality of β, i.e., that $\dot\beta(t) = \dot\beta_1(t)X_1(\beta(t)) + \dot\beta_2(t)X_2(\beta(t))$. By the coordinate form (7.11) of the left-invariant frame, we need to check that

$$\dot\beta_{12} = \frac{1}{2}(\beta_1\dot\beta_2 - \beta_2\dot\beta_1) \quad \text{and} \tag{7.17}$$
$$\dot\beta_{112} = \frac{1}{12}\beta_1^2\dot\beta_2 - \Big(\frac{1}{12}\beta_1\beta_2 + \frac{1}{2}\beta_{12}\Big)\dot\beta_1. \tag{7.18}$$

From the given explicit form of β, we compute the derivatives

$$\dot\beta_1 = -\frac{2\sinh(t)}{\cosh(t)^2},$$
$$\dot\beta_2 = 2(1 - \tanh(t)^2) - 1 = 1 - 2\tanh(t)^2,$$
$$\dot\beta_{12} = \frac{\cosh(t) - t\sinh(t)}{\cosh(t)^2},$$
$$\dot\beta_{112} = \frac{2}{3\cosh(t)^2} - \frac{\cosh(t) - 2t\sinh(t)}{3\cosh(t)^3} = \frac{\cosh(t) + 2t\sinh(t)}{3\cosh(t)^3}.$$

Expanding the right hand side $\frac{1}{2}(\beta_1\dot\beta_2 - \beta_2\dot\beta_1)$ of (7.17), we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{2}{\cosh(t)}\Big(1-\frac{2\sinh(t)^2}{\cosh(t)^2}\Big) - \Big(\frac{2\sinh(t)}{\cosh(t)}-t\Big)\Big(-\frac{2\sinh(t)}{\cosh(t)^2}\Big)\right)\\
&= \frac{1}{\cosh(t)} - \frac{2\sinh(t)^2}{\cosh(t)^3} + \frac{2\sinh(t)^2}{\cosh(t)^3} - \frac{t\sinh(t)}{\cosh(t)^2}\\
&= \frac{\cosh(t)-2t\sinh(t)}{\cosh(t)^2},
\end{aligned}$$

which is exactly $\dot\beta_{12}$. Similarly, expanding the right hand side $\frac{1}{12}\beta_1^2\dot\beta_2 - \Big(\frac{1}{12}\beta_1\beta_2 + \frac{1}{2}\beta_{12}\Big)\dot\beta_1$ of (7.18), we get

$$\begin{aligned}
&\frac{1}{12}\frac{4}{\cosh(t)^2}\Big(1-\frac{2\sinh(t)^2}{\cosh(t)^2}\Big)\\
&-\Big(\frac{1}{12}\frac{2}{\cosh(t)}\Big(\frac{2\sinh(t)}{\cosh(t)}-t\Big)+\frac{1}{2}\frac{t}{\cosh(t)}\Big)\Big)\Big(-\frac{2\sinh(t)}{\cosh(t)^2}\Big)\\
&= \frac{1}{3\cosh(t)^2} - \frac{2\sinh(t)^2}{3\cosh(t)^4} + \frac{2\sinh(t)^2}{3\cosh(t)^4} + \frac{2t\sinh(t)}{3\cosh(t)^3}\\
&= \frac{\cosh(t)+2t\sinh(t)}{3\cosh(t)^3},
\end{aligned}$$

which is exactly $\dot\beta_{112}$, proving horizontality of the curve β.

Finally, we verify that β satisfies the ODE (7.15). Once again, expanding the right hand sides we get

$$-\frac{1}{2}\beta_1\beta_2 - \beta_{12} = -\frac{1}{2}\frac{2}{\cosh(t)}\Big(2\frac{\sinh(t)}{\cosh(t)}-t\Big) - \frac{t}{\cosh(t)} = -\frac{2\sinh(t)}{\cosh(t)^2} = \dot\beta_1$$

and

$$\frac{1}{2}\beta_1^2 - 1 = \frac{1}{2}\frac{4}{\cosh(t)^2} - 1 = \frac{2}{\cosh(t)^2} - 1 = 1 - 2\tanh(t)^2 = \dot\beta_2.$$

q.e.d.

From the explicit form of the infinite geodesic β we can deduce two properties stronger than that of Theorem 1.5: its horizontal projection is asymptotic to a line and the curve itself is in a finite neighborhood of a line.

Proposition 7.19. *Let $L : \mathbb{R} \to E$, $L(t) = \exp(-tX_2)$, which is the abnormal line in the Engel group, and let $\beta : \mathbb{R} \to E$ be the infinite geodesic of Lemma 7.16. Then*

$$\lim_{t\to\infty} d(\beta(t), \exp(\tfrac{2}{3}X_{112})L(t-2)) = 0 \quad \textit{and}$$
$$\lim_{t\to-\infty} d(\beta(t), \exp(-\tfrac{2}{3}X_{112})L(t+2)) = 0.$$

Proof. To prove the claim, we will consider the distances
$$d(\exp(bX_{112})\exp(-(t+a)X_2),\beta(t)),$$
where $a,b\in\mathbb{R}$ are some constants. This distance is zero exactly when the product
$$z(t)=(0,t+a,0,-b)\cdot\beta(t)$$
vanishes.

By the group law (7.10) and the explicit form of the components given in Lemma 7.16, we see that the components of the product $z(t)$ are
$$\begin{aligned}
z_1(t)&=\beta_1(t)=\frac{2}{\cosh(t)},\\
z_2(t)&=\beta_2(t)+t+a=2\tanh(t)+a,\\
z_{12}(t)&=\beta_{12}(t)-\frac{1}{2}(t+a)\beta_1(t)=-\frac{a}{\cosh(t)}\quad\text{and}\\
z_{112}(t)&=\beta_{112}(t)+\frac{1}{12}(t+a)\beta_1(t)^2-b=\frac{2}{3}\tanh(t)+\frac{a}{3\cosh(t)^2}-b.
\end{aligned}$$
From the above we deduce that
$$\lim_{t\to\infty}z(t)=(0,2+a,0,2/3-b)\quad\text{and}\quad\lim_{t\to-\infty}z(t)=(0,-2+a,0,-2/3-b).$$
and the claim of the proposition follows. q.e.d.

Corollary 7.20. *Let $L:\mathbb{R}\to E$, $L(t)=\exp(-tX_2)$, which is the abnormal line in the Engel group, and let $\beta:\mathbb{R}\to E$ be the infinite geodesic of Lemma 7.16. Then*
$$\lim_{t\to\pm\infty}d(\pi\circ\beta(t),\pi\circ L(t))=0\quad\textit{and}\quad\sup_{t\in\mathbb{R}}d(\beta(t),L)<\infty.$$

Proof. Since the horizontal projections of the elements $\exp(\pm\frac{2}{3}X_{112})$ are zero, the lines in Proposition 7.19 have the same horizontal projection as the abnormal line L, and the claim $\lim_{t\to\pm\infty}d(\pi\circ\beta(t),\pi\circ L(t))=0$ follows.

On the other hand, the elements $\exp(\pm\frac{2}{3}X_{112})$ are also in the center of the Engel group, so for all $t\in\mathbb{R}$ we have
$$\begin{aligned}
d(L(t),\exp(\tfrac{2}{3}X_{112})L(t-2))&\le d(L(t),L(t-2))\\
&\quad+d(L(t-2),\exp(\tfrac{2}{3}X_{112})L(t-2))\\
&=2+d(1_E,\exp(\tfrac{2}{3}X_{112})).
\end{aligned}$$
Thus Proposition 7.19 implies that
$$\begin{aligned}
\sup_{t\in\mathbb{R}_+}d(\beta(t),L(t))\le\sup_{t\in\mathbb{R}_+}&\,d(\beta(t),\exp(\tfrac{2}{3}X_{112})L(t-2))\\
&+d(\exp(\tfrac{2}{3}X_{112})L(t-2),L(t))<\infty.
\end{aligned}$$

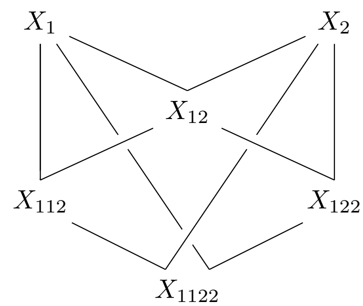

Figure 2. Diagram of relations in the step 4 Lie algebra $\mathfrak{g}$ with the Engel Lie algebra as a quotient.

Similarly using the triangle inequality through $\exp(-\frac{2}{3}X_{112})L(t+2)$ instead of $\exp(\frac{2}{3}X_{112})L(t-2)$, we see that $\sup_{t\in\mathbb{R}_-} d(\beta(t), L(t)) < \infty$, proving the claim. q.e.d.

7.3. Lift of the infinite non-line geodesic to step 4. We shall next show that Theorem 1.5 cannot be improved to say that every sub-Riemannian geodesic is at a finite distance from a lower rank subgroup. Although by Corollary 7.20, this stronger claim is true for the Engel group, the claim is no longer true for the lift of the geodesic β from Lemma 7.16 to a specific Carnot group of rank 2 and step 4.

We will prove the claim by showing that the mismatched limits

$$\lim_{t\to\infty} \beta_{112}(t) = \frac{2}{3} \neq -\frac{2}{3} = \lim_{t\to-\infty} \beta_{112}(t)$$

will cause the lift of β to have different lines as asymptotes as $t \to \infty$ and as $t \to -\infty$ (Proposition 7.25). The claim will then follow from Lemma 7.5, where we proved that the only lines a finite distance apart are right translations of one another.

The specific Carnot group G where we will consider a lift of the Engel geodesic β is the one whose Lie algebra $\mathfrak{g}$ has the basis

$$X_1, X_2, X_{12}, X_{112}, X_{122}, X_{1122},$$

whose only non-zero commutators are (see Figure 2 for a visual description)

$$[X_1, X_2] = X_{12}, \qquad [X_1, X_{12}] = X_{112},$$
$$[X_{12}, X_2] = X_{122}, \qquad [X_1, X_{122}] = [X_{112}, X_2] = X_{1122}.$$

The Lie algebra of the Engel group is a quotient of $\mathfrak{g}$ by the ideal generated by X_{122}, so the Engel group is the quotient of G by the subgroup $H = \exp(\mathrm{span}\{X_{122}, X_{1122}\})$. The metric on G is the sub-Riemannian metric such that the projection $\pi_E : G \to E = G/H$ to the Engel group is a submetry.

Let $\beta : \mathbb{R} \to E$ be the geodesic in the Engel group E given in Lemma 7.16. In exponential coordinates on the Engel group, $\beta(0) = (2,0,0,0)$, so for any initial point $x_0 = (2,0,0,0,x_{122},x_{1122}) \in G$ there exists a horizontal lift of β to G starting from x_0. Let $\alpha : \mathbb{R} \to G$ be the horizontal lift with the initial point $\alpha(0) = (2,0,0,0,2/3,0)$. As with $\beta_1(0) = 2$, the initial coordinate $\alpha_{122}(0) = 2/3$ will simplify the asymptotic behavior. Since the projection $\pi_E : G \to E$ is a submetry and $\pi_E \circ \alpha = \beta$ is an infinite geodesic, the curve α is an infinite geodesic in G.

To study the lift α, we will work in exponential coordinates. The group law is once again given by the BCH formula, which in a nilpotent Lie algebra of step 4 takes the form (for computation of the coefficients, see e.g. [**Var84**, 2.15])

$$\begin{aligned}\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [[X,Y],Y]) \\ + \frac{1}{24}[X,[[X,Y],Y]].\end{aligned}$$

In the first four coordinates, the group law $z = x \cdot y$ is the same as in the Engel group, so the components $z_1, z_2, z_{12}, z_{112}$ are given by (7.10). In the last two coordinates, we have

$$\begin{aligned} z_{122} &= x_{122} + y_{122} + \frac{1}{2}(x_{12}y_2 - x_2y_{12}) \\ &\quad + \frac{1}{12}(x_1y_2^2 - x_1x_2y_2 - x_2y_1y_2 + x_2^2y_1), \\ z_{1122} &= x_{1122} + y_{1122} + \frac{1}{2}(x_1y_{122} - x_{122}y_1 - x_2y_{112} + x_{112}y_2) \\ &\quad - \frac{1}{6}(x_1x_2y_{12} + x_{12}y_1y_2) \\ &\quad + \frac{1}{12}(x_1x_{12}y_2 + x_1y_2y_{12} + x_2x_{12}y_1 + x_2y_1y_{12}) \\ &\quad + \frac{1}{24}(x_1^2y_2^2 - x_2^2y_1^2). \end{aligned} \tag{7.21}$$

The left-invariant extensions of the horizontal vectors X_1 and X_2 are

$$\begin{aligned} X_1(x) &= \partial_1 - \frac{1}{2}x_2\partial_{12} - (\frac{1}{12}x_1x_2 + \frac{1}{2}x_{12})\partial_{112} + \frac{1}{12}x_2^2\partial_{122} \\ &\quad + (\frac{1}{12}x_{12}x_2 - \frac{1}{2}x_{122})\partial_{1122}, \\ X_2(x) &= \partial_2 + \frac{1}{2}x_1\partial_{12} + \frac{1}{12}x_1^2\partial_{112} - (\frac{1}{12}x_1x_2 - \frac{1}{2}x_{12})\partial_{122} \\ &\quad + (\frac{1}{12}x_1x_{12} + \frac{1}{2}x_{112})\partial_{1122}. \end{aligned} \tag{7.22}$$

Lemma 7.23. *In exponential coordinates, the second coordinate of degree 3 of $\alpha : \mathbb{R} \to G$ is*

$$\alpha_{122}(t) = \frac{t^2+4}{6\cosh(t)} + \frac{t\sinh(t)}{3\cosh(t)^2}.$$

Proof. By the explicit form of the left-invariant frame given in (7.22), we need to show that the given expression for α_{122} satisfies both the horizontality condition

$$\dot\alpha_{122} = \dot\alpha_1 X_1(\alpha) + \dot\alpha_2 X_2(\alpha) = \frac{1}{12}\alpha_2^2\dot\alpha_1 - (\frac{1}{12}\alpha_1\alpha_2 - \frac{1}{2}\alpha_{12})\dot\alpha_2 \tag{7.24}$$

and the initial condition $\alpha_{122}(0) = \frac{2}{3}$. The initial condition is immediately verified, since $\alpha_{122}(0) = \frac{4}{6\cosh(0)} = \frac{2}{3}$.

Since α and β agree in the first four coordinates, we get by Lemma 7.16 that

$$\begin{aligned}\frac{1}{12}\alpha_2^2\dot\alpha_1 &= \frac{1}{12}\Big(2\frac{\sinh(t)}{\cosh(t)} - t\Big)^2\Big(-\frac{2\sinh(t)}{\cosh(t)^2}\Big)\\ &= -\frac{2\sinh(t)^3}{3\cosh(t)^4} + \frac{2t\sinh(t)^2}{3\cosh(t)^3} - \frac{t^2\sinh(t)}{6\cosh(t)^2},\end{aligned}$$

$$\begin{aligned}-\frac{1}{12}\alpha_1\alpha_2\dot\alpha_2 &= -\frac{1}{12}\frac{2}{\cosh(t)}\Big(\frac{2\sinh(t)}{\cosh(t)} - t\Big)\Big(1 - \frac{2\sinh(t)^2}{\cosh(t)^2}\Big)\\ &= -\frac{\sinh(t)}{3\cosh(t)^2} + \frac{2\sinh(t)^3}{3\cosh(t)^4} + \frac{t}{6\cosh(t)} - \frac{t\sinh(t)^2}{3\cosh(t)^3},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\alpha_{12}\dot\alpha_2 &= \frac{1}{2}\frac{t}{\cosh(t)}\Big(1 - \frac{2\sinh(t)^2}{\cosh(t)^2}\Big)\\ &= \frac{t}{2\cosh(t)} - \frac{t\sinh(t)^2}{\cosh(t)^3}.\end{aligned}$$

Summing up the above, we get

$$\frac{1}{12}\alpha_2^2\dot\alpha_1 - (\frac{1}{12}\alpha_1\alpha_2 - \frac{1}{2}\alpha_{12})\dot\alpha_2 = \frac{2t}{3\cosh(t)^3} - \frac{(t^2+2)\sinh(t)}{6\cosh(t)^2}.$$

On the other hand, by differentiating the given expression for α_{122}, we also get

$$\frac{d}{dt}\Big(\frac{t^2+4}{6\cosh(t)} + \frac{t\sinh(t)}{3\cosh(t)^2}\Big) = \frac{2t}{3\cosh(t)^3} - \frac{(t^2+2)\sinh(t)}{6\cosh(t)^2},$$

so the horizontality condition (7.24) is satisfied. q.e.d.

Proposition 7.25. *Let $L_\pm(t) = \exp(-(tX_2 \pm \frac{2}{3}tX_{1122}))$. Then*

$$\sup_{t\in\mathbb{R}_+} d(\alpha(t), L_+(t)) < \infty \quad \textit{and} \quad \sup_{t\in\mathbb{R}_-} d(\alpha(t), L_-(t)) < \infty.$$

Proof. As in Proposition 7.19, we compute the distances $d(\alpha(t), L_\pm(t))$ directly by the considering the products $L_\pm(t)^{-1}\alpha(t)$. Since the lines L_+ and L_- only differ by the sign of $\frac{2}{3}tX_{1122}$, we will combine the computations. That is, we will consider the product

$$z(t) := \exp(tX_2 \pm \tfrac{2}{3}tX_{1122})\alpha(t).$$

The group law in the first four coordinates is exactly the group law of the Engel group (7.10), so the first four components $z_1, z_2, z_{12}, z_{112}$ are bounded by Corollary 7.20. It remains to consider the components z_{122} and z_{1122}.

By the group law (7.21), we have

$$z_{122}(t) = \alpha_{122}(t) - \frac{1}{2}t\alpha_{12}(t) - \frac{1}{12}t\alpha_1(t)\alpha_2(t) + \frac{1}{12}t^2\alpha_1(t) \quad \text{and}$$
$$z_{1122}(t) = \alpha_{1122}(t) \pm \frac{2}{3}t - \frac{1}{2}t\alpha_{112}(t) + \frac{1}{12}t\alpha_1(t)\alpha_{12}(t) - \frac{1}{24}t^2\alpha_1(t)^2.$$

By the explicit expressions given in Lemma 7.16, we see that the components $\alpha_1 = \beta_1$ and $\alpha_{12} = \beta_{12}$ are both exponentially asymptotically vanishing, i.e., for any polynomial $P : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{t\to\pm\infty} P(t)\alpha_1(t) = \lim_{t\to\pm\infty} P(t)\alpha_{12}(t) = 0.$$

Therefore there exists a constant $C > 0$ such that

$$|z_{122}(t)| \le |\alpha_{122}(t)| + C \quad \text{and}$$
$$|z_{1122}(t)| \le \left|\alpha_{1122}(t) \pm \frac{2}{3}t - \frac{1}{2}t\alpha_{112}(t)\right| + C. \tag{7.26}$$

By the explicit form in Lemma 7.23, we see that α_{122} is bounded, so the same is true for z_{122}. For z_{1122}, we will consider the term

$$w(t) := \alpha_{1122}(t) \pm \frac{2}{3}t - \frac{1}{2}t\alpha_{112}(t)$$

separately for $t > 0$ and $t < 0$.

Instead of explicitly computing α_{1122}, we will consider the derivative $\dot{w}$. Since α is a horizontal curve, from the explicit form (7.22) of the left-invariant frame, we get the identity

$$\dot{\alpha}_{1122} = (\frac{1}{12}\alpha_{12}\alpha_2 - \frac{1}{2}\alpha_{122})\dot{\alpha}_1 + (\frac{1}{12}\alpha_1\alpha_{12} + \frac{1}{2}\alpha_{112})\dot{\alpha}_2.$$

By the explicit expressions given in Lemmas 7.16 and 7.23, we see that as $t \to \pm\infty$ the terms $\alpha_1, \alpha_{12}, \alpha_{122}, \dot{\alpha}_{112}$ and $\dot{\alpha}_2 + 1$, are all exponentially vanishing. It follows that

$$\dot{w}(t) = -\alpha_{112}(t) \pm \frac{2}{3} + \epsilon(t), \tag{7.27}$$

where $\epsilon : \mathbb{R} \to \mathbb{R}_+$ is some smooth function such that $\epsilon(t) = O(e^{-|t|})$ as $t \to \pm\infty$.

Finally, we observe that as $t \to \infty$, $\alpha_{112}(t) - \frac{2}{3} = O(e^{-t})$, and as $t \to -\infty$, $\alpha_{112}(t) + \frac{2}{3} = O(e^{t})$. Therefore from (7.27) we conclude that as $t \to \infty$ we have

$$\left|\alpha_{1122}(t) + \frac{2}{3}t - \frac{1}{2}t\alpha_{112}(t)\right| \le \int_0^t \left|-\alpha_{112}(s) + \frac{2}{3} + \epsilon(s)\right| ds = O(e^{-t}).$$

It follows from (7.26) that also the final coordinate of $L_+(t)^{-1}\alpha(t)$ is bounded on $\mathbb{R}_+$. Thus the product $L_+(t)^{-1}\alpha(t)$ is bounded on $\mathbb{R}_+$.

Similarly for $t \to -\infty$ we conclude that

$$\left|\alpha_{1122}(t) - \frac{2}{3}t - \frac{1}{2}t\alpha_{112}(t)\right| = O(e^{t}),$$

from which it follows that the product $L_-(t)^{-1}\alpha(t)$ is bounded on $\mathbb{R}_-$, proving the claim. q.e.d.

Corollary 7.28. *Let $L : \mathbb{R} \to G$ be any line. Then $d_H(\alpha(\mathbb{R}), L(\mathbb{R})) = \infty$.*

Proof. The corollary follows from combining Lemma 7.5 and Proposition 7.25. Suppose there existed a line $L \subset G$ such that $d_H(\alpha(\mathbb{R}), L(\mathbb{R})) < \infty$. Then also $d_H(\pi \circ \alpha(\mathbb{R}), \pi \circ L(\mathbb{R})) \le M$, so from the explicit form of the horizontal components of α given in Lemma 7.16, we see that $\pi \circ L$ must be parallel to the x_2-axis.

Up to reparametrizing L we can then assume that $\pi \circ L(t) = (C, -t)$ for some $C \in \mathbb{R}$. In particular, we have

$$d(\alpha(t), L(s)) \ge d(\pi \circ \alpha(t), \pi \circ L(s)) \ge |t - s| - 2.$$

Then by Lemma 7.2, since $d_H(\alpha(\mathbb{R}), L(\mathbb{R})) < \infty$, we have that also $\sup_t d(\alpha(t), L(t)) < \infty$. In particular $d_H(\alpha(\mathbb{R}_+), L(\mathbb{R}_+)) < \infty$ and $d_H(\alpha(\mathbb{R}_-), L(\mathbb{R}_-)) < \infty$.

Let $L_\pm(t) = \exp(tY_\pm)$ be the lines of Proposition 7.25. Proposition 7.25 and the triangle inequality for the Hausdorff distance imply that

$$d_H(L(\mathbb{R}_+), L_+(\mathbb{R}_+)) \le d_H(L(\mathbb{R}_+), \alpha(\mathbb{R}_+)) + d_H(\alpha(\mathbb{R}_+), L_-(\mathbb{R}_+)) < \infty$$

and similarly that $d_H(L(\mathbb{R}_-), L_2(\mathbb{R}_-)) < \infty$. By applying Lemma 7.5 to both halves of the line L, we get the existence of constants $c_\pm > 0$ such that

$$X = c_- \operatorname{Ad}_{g^{-1}} Y_- = c_+ \operatorname{Ad}_{g^{-1}} Y_+,$$

where X and g are such that $L(t) = g\exp(tX)$. This implies that Y_+ and Y_- are linearly dependent, which is a contradiction. q.e.d.

Corollary 7.28 shows that $\alpha : \mathbb{R} \to G$ is a geodesic that is not in a finite neighborhood of any line, showing that the claim of Theorem 1.5 cannot hold without considering the projection $\pi : G \to G/[G, G]$. Still Conjecture 7.1 may be true.

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