GRADINGS FOR NILPOTENT LIE ALGEBRAS

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ABSTRACT. We present a constructive approach to torsion-free gradings of Lie algebras. Our main result is the computation of a maximal grading. Given a Lie algebra, using its maximal grading we enumerate all of its torsion-free gradings as well as its positive gradings. As applications, we classify gradings in low dimension, we consider the enumeration of Heintze groups, and we give methods to find bounds for non-vanishing $\ell^{q,p}$ cohomology.

CONTENTS

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1. Introduction

1.1. **Overview.**

A *grading* of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in S} V_\alpha \tag{1}$$

indexed by some set S in such a way that for each pair $\alpha, \beta \in S$ there exists $\gamma \in S$ such that

$$[V_\alpha, V_\beta] \subset V_\gamma.$$

In this paper, we will focus on Lie algebras defined over fields of characteristic zero and gradings indexed over torsion-free abelian groups, where the element γ is given by $\gamma = \alpha + \beta$.

An important example of a Lie algebra grading is the so called *maximal grading* (also known as *fine grading*), that is a grading that does not admit any proper refinement into smaller subspaces V_α. A classical example of such a maximal grading is the Cartan decomposition, which plays a fundamental role in representation theory and the classification of semisimple Lie algebras over $\mathbb{C}$, see for example [Hum78]. There has been a growing interest in the study of (maximal) gradings of semisimple Lie algebras since the paper [PZ89], see the survey [Koc09] or the monograph [EK13] for an overview. Moreover, a classification of maximal gradings of simple classical Lie algebras over algebraically closed fields of characteristic zero can be found in [Eld10].

Regarding nilpotent Lie algebras over algebraically closed fields of characteristic zero, an in depth study of maximal gradings over torsion-free abelian groups was carried out in [Fav73]. One of the main results in [Fav73] is that, given a nilpotent Lie algebra $\mathfrak{g}$ of nilpotency step s and with abelianization $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of dimension r, there are only finitely many torsion-free maximal gradings, up to automorphisms of the free nilpotent Lie algebra of step s with r generators. This finiteness in the number of maximal gradings is in contrast with the existence of an uncountable number of non-isomorphic nilpotent Lie algebras in dimension 7 and higher.

There are two other special types of gradings of particular interest in the case of nilpotent Lie algebras: *positive gradings* and *stratifications* (also called *Carnot gradings*). A positive grading is a grading indexed over the reals such that in the direct sum decomposition (1) all the non-zero spaces V_α have positive indices $\alpha > 0$. A stratification is a positive grading for which V_1 generates $\mathfrak{g}$ as a Lie algebra.

Lie algebras with a stratification are the Lie algebras of Carnot groups. These groups have played a central role in the fields of geometric analysis, geometric measure theory, and large scale geometry, see [LD17] for a long list of references.

Positive gradings are important within the study of homogeneous spaces, as they appear directly in characterizations of such spaces.

First, any negatively curved homogeneous Riemannian manifold is a *Heintze group* $G \rtimes \mathbb{R}$ [Hei74], where G is a nilpotent Lie group and the action of $\mathbb{R}$ on G is given by the one-parameter family of automorphisms associated with a positive grading of G. Second, any connected locally compact group that admits a contracting automorphism is a positively gradable Lie group [Sie86]. In this latter result, the group structure and contracting automorphism may also be replaced by a metric structure and a dilation, see [CKLD$^+$17].

Another active area of research that contains several open problems relating to positively gradable Lie groups is the quasi-isometric classification of locally compact groups. A survey on the topic can be found in [Cor18]. For instance, it is not known whether there exists a non-stratifiable positively gradable Lie group that is quasi-isometric to its asymptotic cone, nor whether all large-scale contractible groups are positively gradable, see [Cor19, Question 7.9]. The quasi-isometric classification is open also for Heintze groups, see [CPS17] for some known results.

1.2. **Main results.** In all of the following statements, let $\mathfrak{g}$ be a finite dimensional Lie algebra defined in terms of its structure coefficients and let F be the base field of $\mathfrak{g}$. That is, we assume we have a fixed basis $X_1, \dots, X_n$ of $\mathfrak{g}$ and a family of coefficients $\{c_{ij}^k \in F : i, j, k \in \{1, \dots, n\}\}$ such that the Lie bracket is defined as

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

Our main result is the following.

Theorem 1.1. *Suppose the base field F is algebraically closed. Then there exists an algorithm that constructs a maximal grading of $\mathfrak{g}$.*

We also give explicit constructions for stratifications and positive gradings.

Theorem 1.2. *There exists an algorithm that constructs a stratification of $\mathfrak{g}$ or determines that one does not exist.*

Theorem 1.3. *Suppose the base field F is algebraically closed. Then there exists an algorithm that constructs a positive grading of $\mathfrak{g}$ or determines that one does not exist.*

Theorem 1.2 and Theorem 1.3 are constructive versions of the characterizations of stratifiability in [Cor16, Lemma 3.10] and existence of a positive grading in [Cor16, Proposition 3.22].

Using Theorem 1.1, we are able to enumerate all torsion-free gradings.

Theorem 1.4. *Suppose the base field F is algebraically closed. Then there exists an algorithm to compute a finite collection of gradings containing up to equivalence all the torsion-free gradings of $\mathfrak{g}$.*

A torsion-free grading is a grading that can be indexed over a torsion-free abelian group, and gradings are considered equivalent if there is an automorphism of the Lie algebra mapping layers of one grading to layers of the other. The precise definitions can be found in Section 2. The finite set we construct in Theorem 1.4 will in general contain redundant gradings, i.e., there may exist equivalent gradings in the collection. We eliminate this redundancy in the case of nilpotent Lie algebras of dimension up to 6 to find a complete classification up to equivalence of torsion-free gradings.

For applications related to positive gradings, we also give a method to extract from the complete list of Theorem 1.4 of all torsion-free gradings those that admit a positive realization, i.e., can be indexed over the positive reals:

Theorem 1.5. *Let $\mathfrak{g} = \bigoplus_{\alpha\in S} V_\alpha$ be a grading of $\mathfrak{g}$.*

(i) There exists an algorithm that constructs a positive realization of the grading or determines that one does not exist.

(ii) If S is a finitely generated abelian group, then there exists an algorithm that constructs a positive realization such that the reindexing $S \to \mathbb{R}$ of layers is a homomorphism, or determines that one does not exist.

We also give two applications of the enumeration of positive gradings obtained from the above results. First, we show that all non-equivalent positive gradings define non-isomorphic Heintze groups, see Proposition 4.7. In this way we are able to enumerate diagonal Heintze groups. Thus we give methods to tackle the problem of finding all Heintze groups with prescribed nilradical, which is a question already posed by Heintze in [Hei74].

Second, the enumeration of positive gradings gives a method to find better estimates for the non-vanishing of the $\ell^{q,p}$ cohomology of a nilpotent Lie group, which is a quasi-isometry invariant.

1.3. **Structure of the paper.** In Section 2 we recall various definitions and terminology related to gradings. The core concepts of realization, push-forward, and equivalence are defined in Subsection 2.1 and universal realizations are recalled in Subsection 2.2. Subsection 2.3 recalls how to study torsion-free gradings of a Lie algebra $\mathfrak{g}$ in terms of subtori of the derivation algebra $\mathrm{der}(\mathfrak{g})$. Maximal gradings and their universal property are covered in Subsection 2.4. Subsection 2.5 reduces Theorem 1.4 on enumeration of gradings to proving Theorem 1.1 on algorithmic construction of a maximal grading.

In Section 3 we give the remaining constructions for our main results. Subsection 3.1 covers Theorem 1.2 on stratifiability. Subsection 3.2 covers Theorem 1.3 and Theorem 1.5 on positive gradings. An alternate approach to deciding the existence of a positive realization is also described in Appendix A. Subsection 3.3 covers Theorem 1.1 on maximal gradings.

In Section 4 we give various applications of gradings to the study of Lie algebras and Lie groups. Subsection 4.1 shows how to use the maximal grading of a Lie algebra as a tool to detect decomposability of a Lie algebra, and how to reduce the dimensionality of the problem of deciding whether two Lie algebras are isomorphic. In Subsection 4.2 we classify up to equivalence the gradings of low dimensional nilpotent Lie algebras over $\mathbb{C}$. In Subsection 4.3 we cover the results on enumeration of Heintze groups. Finally, in Subsection 4.4 we present the method to find improved bounds for the non-vanishing of the $\ell^{q,p}$ cohomology.

2. Gradings

The contents of this section can, up to some modifications, be found in [Koc09, Section 3-4]. We nonetheless give here a self-contained presentation to better fit our constructive approach.

2.1. **Gradings and equivalences.**

In this section we define some key notions related to gradings of Lie algebras, including equivalence, push-forwards and coarsenings. We also make a distinction between two different notions of grading, with the difference being whether the indexing plays a role or not.

Definition 2.1. A *grading* of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in S} V_\alpha$ such that for each $\alpha, \beta \in S$ either $[V_\alpha, V_\beta] = 0$ or there exists a unique $\gamma \in S$ such that $[V_\alpha, V_\beta] \subset V_\gamma$. When S is an abelian group A such that the unique element γ is given by $\gamma = a + b$, we say that the grading $\mathcal{V}$ is *over* A, or that $\mathcal{V}$ is an A-*grading.* In this case, A is the *grading group* of the grading $\mathcal{V}$.

The subspaces V_α are called the *layers* of the grading $\mathcal{V}$ and the elements $\alpha \in S$ such that $V_\alpha \neq 0$ are called the *weights* of $\mathcal{V}$. We will usually denote the set of weights by Ω. A basis of $\mathfrak{g}$ is said to be *adapted to* $\mathcal{V}$ if every element of the basis is contained in some layer of $\mathcal{V}$.

Definition 2.2. Suppose the indexing set S of a grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in S} V_\alpha$ can be embedded into an abelian group A such that $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$ for all $\alpha, \beta \in A$, where we define $V_\alpha = 0$ for $\alpha \notin S$. Then the resulting A-grading is called a *realization* of the grading $\mathcal{V}$.

Definition 2.3. A grading is called *torsion-free* if it admits a realization over a torsion-free (abelian) group.

In this paper, the notation $\langle X \rangle$ always refers to the span of X in the appropriate sense.

Example 2.4. Consider the 6-dimensional Lie algebra spanned by the vectors $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ with the non-trivial bracket relations

$$[X_1, Y_1] = Z_1 \qquad [X_2, Y_2] = Z_2.$$

The subspace decomposition

$$V_a = \langle X_1 \rangle, \qquad V_b = \langle X_2 \rangle, \qquad V_c = \langle Y_1, Z_2 \rangle, \qquad V_d = \langle Z_1, Y_2 \rangle$$

defines a grading. It can realized over $\mathbb{Z}^2$ with the embedding

$$a \mapsto (1,0), \qquad b \mapsto (-1,0), \qquad c \mapsto (0,1), \qquad d \mapsto (1,1).$$

Definition 2.5. Let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ be an A-grading for some abelian group A. Given an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$, an abelian group B and a homomorphism $f : A \to B$, we define the *push-forward grading* $f_*\Phi(\mathcal{V}) : \mathfrak{g} = \bigoplus_{\beta \in B} W_\beta$ over B, where

$$W_\beta = \bigoplus_{\alpha \in f^{-1}(\beta)} \Phi(V_\alpha).$$

When $\Phi = \mathrm{Id}$, we simply denote $f_* \mathrm{Id}(\mathcal{V}) = f_*\mathcal{V}$.

It is readily checked that the push-forward grading is indeed a B-grading in the sense of Definition 2.1.

Definition 2.6. Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in S_1} V_\alpha$ and $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta \in S_2} W_\beta$ be two gradings. If for every $\alpha \in S_1$ there exists $\beta \in S_2$ such that $V_\alpha \subset W_\beta$, then we say that $\mathcal{V}$ is a *refinement* of $\mathcal{W}$, and that $\mathcal{W}$ is a *coarsening* of $\mathcal{V}$.

Remark 2.7. If $\mathcal{W} = f_*\mathcal{V}$ for some homomorphism f, then $\mathcal{W}$ is a coarsening of $\mathcal{V}$. Such a map f is injective on the weights if and only if $\mathcal{V}$ and $\mathcal{W}$ are realizations of the same grading.

There are several different notions of equivalence of gradings in the literature. The two that we shall use are distinguished as *equivalence* and *group-equivalence* in [Koc09]. For brevity, we will refer to both notions as equivalence. Stated in terms of push-forwards, the group-equivalence notion of [Koc09] takes the following form:

Definition 2.8. An A-grading $\mathcal{V}$ and a B-grading $\mathcal{W}$ are said to be *equivalent* if there exist an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ and a group isomorphism $f : A \to B$ such that $\mathcal{W} = f_*\Phi(\mathcal{V})$.

For gradings that admit realizations, the equivalence notion of [Koc09] can be rephrased through the previous notion as follows.

Definition 2.9. A grading $\mathfrak{g} = \bigoplus_{\alpha \in S_1} V_\alpha$ and a grading $\mathfrak{g} = \bigoplus_{\beta \in S_2} W_\beta$ over arbitrary indexing sets S_1, S_2 are said to be *equivalent* if they admit realizations as an A-grading and a B-grading that are equivalent in the sense of Definition 2.8.

Example 2.10. Consider two gradings $V_1 \oplus V_2$ and $V_1 \oplus V_3$ of $\mathbb{R}^2$ over $\mathbb{Z}$ with the same one-dimensional layers. The two gradings are equivalent in the sense of Definition 2.9, since the former is a realization of the second by the embedding $\{1,3\} \hookrightarrow \{1,2\} \subset \mathbb{Z}$, but they are not equivalent as $\mathbb{Z}$-gradings in the sense of Definition 2.8 as there does not exist an automorphism of $\mathbb{Z}$ mapping $\{1,3\} \to \{1,2\}$.

In the following lemma we demonstrate that, after possibly shrinking the grading groups, an A-grading and a B-grading are equivalent if and only if they are push-forwards of each other.

Lemma 2.11. *Let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ be an A-grading and $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta \in B} W_\beta$ be a B-grading such that the weights of $\mathcal{V}$ and $\mathcal{W}$ generate the abelian groups A and B, respectively. If there exist homomorphisms $f\colon A \to B$ and $g\colon B \to A$ such that $\mathcal{W} = f_*\mathcal{V}$ and $\mathcal{V} = g_*\mathcal{W}$, then $\mathcal{V}$ and $\mathcal{W}$ are equivalent.*

Proof. Let us denote by Ω_A and Ω_B the sets of weights of $\mathcal{V}$ and $\mathcal{W}$. Notice first that by definition of the push-forward, $f(\Omega_A) = \Omega_B$ and $g(\Omega_B) = \Omega_A$, so both f and g are injective on weights. Moreover, we have for every $\alpha \in \Omega_A$ and $\beta \in \Omega_B$ the correspondence

$$V_\alpha = W_{f(\alpha)} = V_{g(f(\alpha))} \qquad \text{and} \qquad W_\beta = V_{g(\beta)} = W_{f(g(\beta))}.$$

Hence $f\colon \Omega_A \to \Omega_B$ is a bijection and $f^{-1} = g$ on Ω_B. Since Ω_A and Ω_B generate A and B as groups, we get that $f^{-1} = g$ on whole B. □

Notice that the assumption that the weights generate is indeed necessary: for instance, the gradings $\mathbb{R} = V_1$ over $\mathbb{Z}$ and $\mathbb{R} = V_{(1,0)}$ over $\mathbb{Z}^2$ are push-forward gradings of each other, but they are not equivalent.

2.2. **Universal gradings.** We do not in general require that the weights of an A-grading generate the grading group A in order to include e.g. gradings over $A = \mathbb{R}$ in the discussion. Moreover, weights of a grading may have additional relations coming from the ambient group structure, even when the corresponding layers are unrelated, see for instance Example 3.13. To build a satisfactory theory using homomorphisms between grading groups, we consider the notion of an (abelian) universal realization, see [Koc09, Section 3.3].

Definition 2.12. Let $\mathcal{V}$ be a grading of $\mathfrak{g}$. A *universal realization* of $\mathcal{V}$ is a realization $\widetilde{\mathcal{V}}$ as an A-grading such that for every realization of $\mathcal{V}$ as a B-grading with B abelian, there exists a unique homomorphism $f\colon A \to B$ such that the B-grading is the push-forward grading $f_*\widetilde{\mathcal{V}}$.

Observe that by Lemma 2.11, the universal realization of a grading is unique up to equivalence.

If a grading admits a realization, then it also admits a universal realization. The universal realization can be constructed by considering the free abelian group generated by the weights and quotienting out the grading relations, as described by the following algorithm.

Algorithm 2.13 (Universal realization)**.** *Input: a grading $\mathcal{V}$ that has a realization. Output: a universal realization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$.*

1. *Let $\Omega = \{\alpha_1, \dots, \alpha_n\}$ be the set of weights of $\mathcal{V}$ and let $B = \{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{Z}^n$. Set $R = \emptyset$.*
2. *Repeat for all pairs $\alpha_i, \alpha_j \in \Omega$: If $0 \neq [V_{\alpha_i}, V_{\alpha_j}] \subseteq V_{\alpha_k}$ for some $\alpha_k \in \Omega$, extend R by $e_i + e_j - e_k$.*
3. *For all $i = 1, \dots, n$, set $\widetilde{V}_{\pi(e_i)} = V_{\alpha_i}$, where $\pi\colon \mathbb{Z}^n \to \mathbb{Z}^n/\langle R\rangle$ is the projection. Return the obtained $\mathbb{Z}^n/\langle R\rangle$-grading.*

Proof of correctness. Consider a realization of $\mathcal{V}$ over an abelian group A and the homomorphism $\phi\colon \mathbb{Z}^n \to A$ defined by $\phi(e_i) = \alpha_i$ for all $1 \leq i \leq n$. Observe that by construction $R \subset \ker(\phi)$. Then the grading $\widetilde{\mathcal{V}}$ is well-defined: if $\pi(e_i) = \pi(e_j)$, then $e_i - e_j \in \langle R\rangle$ and we have $\alpha_i = \phi(e_i) = \phi(e_j) = \alpha_j$. Moreover, the obtained $\mathbb{Z}^n/\langle R\rangle$-grading is a universal realization of $\mathcal{V}$ by the universal property of quotients and arbitrariness of A. ☐

In the rest of the paper we will focus on gradings that admit torsion-free realizations. For such gradings, the universal realizations are gradings over some $\mathbb{Z}^k$, as demonstrated by the following lemma.

Lemma 2.14. *If $\mathcal{V}$ is a torsion-free grading, then the grading group of the universal realization of $\mathcal{V}$ is isomorphic to some $\mathbb{Z}^k$.*

Proof. Let $\widetilde{\mathcal{V}}$ be the universal realization of $\mathcal{V}$. By Algorithm 2.13, $\widetilde{\mathcal{V}}$ is a $\mathbb{Z}^n/\langle R\rangle$-grading for some subset $R \subset \mathbb{Z}^n$. The quotient $\mathbb{Z}^n/\langle R\rangle$ is isomorphic to a group $\mathbb{Z}^k \times G_t$, where G_t is some torsion group.

By assumption there exists a realization of $\mathcal{V}$ as an A-grading with A torsion-free. Since the image of G_t under a homomorphism must vanish in A, we conclude that there are no non-zero weights in G_t. Since a universal realization is generated by its weights, we conclude that $G_t = 0$, and $\widetilde{\mathcal{V}}$ is a $\mathbb{Z}^k$-grading. ☐

The following lemma is a part of [Koc09, Proposition 3.15], and we record it for later usage.

Lemma 2.15. *If a grading $\mathcal{V}$ is a coarsening of a grading $\mathcal{W}$, then every realization of $\mathcal{V}$ is a push-forward grading of the universal realization of $\mathcal{W}$.*

2.3. **Gradings induced by tori.** In this subsection we describe the correspondence between gradings of a Lie algebra $\mathfrak{g}$ and the split tori of its derivation algebra $\mathrm{der}(\mathfrak{g})$. In general, gradings of a Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with algebraic quasitori, see [Koc09, Section 4]. However, in this study we are only interested in cases when $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero and the gradings are over torsion-free abelian groups. In this setting, the characterization of gradings in terms of algebraic quasitori can be reduced to studying algebraic subtori of the derivation algebra $\mathrm{der}(\mathfrak{g})$.

For computational reasons, we will drop the algebraicity requirement for the subalgebras of $\mathrm{der}(\mathfrak{g})$. This means we lose the one-to-one correspondence described in [Koc09], but the less restrictive definition will be sufficient for our purposes. In particular, it will simplify the explicit construction of maximal gradings in terms of tori, see Subsection 3.3.

We start by defining split tori and gradings induced by them in the sense of [Fav73].

Definition 2.16. An abelian subalgebra $\mathfrak{t}$ of semisimple derivations of $\mathfrak{g}$ is called a *torus* of $\mathrm{der}(\mathfrak{g})$. If the torus $\mathfrak{t}$ is diagonalizable over the base field of $\mathfrak{g}$, it is called a *split torus*.

Lemma 2.17. *Let $\mathfrak{t}$ be a split torus of $\mathrm{der}(\mathfrak{g})$ and let $\mathfrak{t}^*$ be its dual as a vector space. For each $\alpha \in \mathfrak{t}^*$ define the subspace*

$$V_\alpha = \{X \in \mathfrak{g} : \delta(X) = \alpha(\delta)X \,\forall \delta \in \mathfrak{t}\}.$$

Then $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} V_\alpha$ is a $\mathfrak{t}^*$-grading.

Proof. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ that diagonalizes $\mathfrak{t}$. Since each vector X_i is an eigenvector of every derivation $\delta \in \mathfrak{t}$, there are well defined linear maps $\alpha_1, \ldots, \alpha_n \in \mathfrak{t}^*$ determined by

$$\delta(X_i) = \alpha_i(\delta)X_i, \quad i = 1, \ldots, n.$$

By construction $X_i \in V_{\alpha_i}$, so the direct sum $\bigoplus_{\alpha\in\mathfrak{t}^*} V_\alpha$ spans all of the Lie algebra $\mathfrak{g}$. The inclusion $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$ follows by linearity from the Leibniz rule $\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$ for all derivations $\delta \in \mathfrak{t}$ and vectors $X \in V_\alpha$ and $Y \in V_\beta$. ☐

Definition 2.18. The $\mathfrak{t}^*$-grading of $\mathfrak{g}$ defined in Lemma 2.17 is called the *grading induced by the split torus* $\mathfrak{t}$.

See Example 3.13 for some gradings induced by tori in the Heisenberg Lie algebra.

For the purposes of Subsection 2.4, we need the following two lemmas. In Lemma 2.19 we link equivalences and push-forwards of gradings to relations between the inducing tori.

Lemma 2.19. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two split tori of $\mathrm{der}(\mathfrak{g})$ with respective induced $\mathfrak{t}_1^*$-grading $\mathcal{V}$ and $\mathfrak{t}_2^*$-grading $\mathcal{W}$.*

(i) If there exists an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\Phi \circ \mathfrak{t}_1 \circ \Phi^{-1} = \mathfrak{t}_2$, then $\mathcal{V}$ and $\mathcal{W}$ are equivalent.

(ii) If $\mathfrak{t}_1 \subset \mathfrak{t}_2$, then there exists a homomorphism f so that $\mathcal{V} = f_*\mathcal{W}$.

Proof. To show (i), suppose that $\mathrm{Ad}_\Phi \mathfrak{t}_1 = \Phi \circ \mathfrak{t}_1 \circ \Phi^{-1} = \mathfrak{t}_2$ for some automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$. Let $f\colon \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ be the linear isomorphism $f = \mathrm{Ad}^*_{\Phi^{-1}}$ given by $f(\alpha)(\delta) = \alpha(\Phi^{-1} \circ \delta \circ \Phi)$. Then

$$\begin{aligned}\Phi(V_\alpha) &= \{\Phi(X) : \delta(X) = \alpha(\delta)X\, \forall \delta \in \mathfrak{t}_1\}\\ &= \{Y : \Phi \circ \delta \circ \Phi^{-1}(Y) = \alpha(\delta)Y\, \forall \delta \in \mathfrak{t}_1\}\\ &= \{Y : \eta(Y) = f(\alpha)(\eta)Y\, \forall \eta \in \mathfrak{t}_2\} = W_{f(\alpha)}.\end{aligned}$$

Hence the gradings $\mathcal{V}$ and $\mathcal{W}$ are equivalent, as claimed.

Regarding (ii), suppose that $\mathfrak{t}_1 \subset \mathfrak{t}_2$. We claim that $\mathcal{V} = g_*\mathcal{W}$ through the restriction map $g\colon \mathfrak{t}_2^* \to \mathfrak{t}_1^*$, $g(\beta) = \beta|_{\mathfrak{t}_1}$. Indeed, fix a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ that diagonalizes the split torus $\mathfrak{t}_2$ (and hence also the subtorus $\mathfrak{t}_1$). Let $\beta_1, \dots, \beta_n \in \mathfrak{t}_2^*$ be the maps defined by $\delta(X_i) = \beta_i(\delta)X_i$ and define $\alpha_i = \beta_i|_{\mathfrak{t}_1}$. By construction $X_i \in W_{\beta_i}$, $X_i \in V_{\alpha_i}$, and $g(\beta_i) = \alpha_i$, proving that $\mathcal{V} = g_*\mathcal{W}$. □

Finally, we observe that any torsion-free grading has a realization induced by a split torus.

Lemma 2.20. *Let $\mathcal{V}$ be a torsion-free grading. Then there exists a split torus $\mathfrak{t}$ whose induced $\mathfrak{t}^*$-grading is a realization of $\mathcal{V}$.*

Proof. Let $\mathcal{V}\colon \mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ be a realization of $\mathcal{V}$ over a torsion-free abelian group A and let A^* be the space of homomorphisms $A \to F$, where F is the base field of $\mathfrak{g}$. By reducing to the subgroup generated by the weights, we may assume A is isomorphic to $\mathbb{Z}^m$ for some $m \geq 1$. For each $\varphi \in A^*$ define the linear map

$$\delta_\varphi\colon \mathfrak{g} \to \mathfrak{g}, \quad \delta_\varphi(X) = \varphi(\alpha)X \quad \forall X \in V_\alpha.$$

We claim that $\mathfrak{t} = \{\delta_\varphi : \varphi \in A^*\}$ is a split torus that induces a realization for $\mathcal{V}$. Indeed, a direct computation shows that all the maps δ_φ are derivations. They are diagonalizable since by construction they are multiples of the identity on each layer V_α. Hence $\mathfrak{t}$ is a split torus.

Let then $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta \in \mathfrak{t}^*} W_\beta$ be the $\mathfrak{t}^*$-grading induced by $\mathfrak{t}$. Denote by Ω the set of weights of $\mathcal{V}$, and define a map $f : \Omega \to \mathfrak{t}^*$ by $f(\alpha)(\delta_\varphi) = \varphi(\alpha)$. Then f is well-defined: if $\varphi, \phi \in A^*$ are such that $\delta_\varphi = \delta_\phi$, then by the definition of $\mathfrak{t}$ we have $\varphi(\alpha) = \phi(\alpha)$ for all weights $\alpha \in \Omega$.

First, we show that $V_\alpha \subset W_{f(\alpha)}$ for every $\alpha \in A$. By the construction of the torus $\mathfrak{t}$, for each $X \in V_\alpha$ we have that

$$\delta_\varphi(X) = \varphi(\alpha)X = f(\alpha)(\delta_\varphi)X \quad \forall \delta_\varphi \in \mathfrak{t}.$$

By the definition of the grading $\mathcal{W}$, we then have $X \in W_{f(\alpha)}$ and so $V_\alpha \subset W_{f(\alpha)}$.

Next, we show that the map f is injective, which would prove that $V_\alpha = W_{f(\alpha)}$ for all $\alpha \in \Omega$ and so $\mathcal{W}$ would be a realization of $\mathcal{V}$, as claimed. Note that since A is isomorphic to $\mathbb{Z}^m$, for every non-zero $\alpha \in A$ there exists a homomorphism $\varphi \in A^*$ such that $\varphi(\alpha) \neq 0$. Therefore, if $\alpha, \alpha' \in \Omega$ are such that $f(\alpha) = f(\alpha')$, then by the construction of the map f we have

$$\varphi(\alpha - \alpha') = \varphi(\alpha) - \varphi(\alpha') = f(\alpha)(\delta_\varphi) - f(\alpha')(\delta_\varphi) = 0$$

for every homomorphism $\varphi \colon A \to F$. So $\alpha = \alpha'$ and f is injective, proving that $\mathcal{W}$ is a realization of $\mathcal{V}$. □

2.4. **Maximal gradings.** We now present the notion of maximal grading using maximal split tori and prove that a maximal grading has the universal property of push-forwards (see Proposition 2.23). The formulation through the derivation algebra will be convenient in the construction of maximal grading in Subsection 3.3. The universal property will be exploited in Subsection 2.5 where we give a method to construct all gradings over torsion-free abelian groups of a Lie algebra from a given maximal grading.

Definition 2.21. Let $\mathfrak{g}$ be a Lie algebra. A *maximal grading* of $\mathfrak{g}$ is the universal realization of the grading induced by a maximal split torus of $\mathrm{der}(\mathfrak{g})$.

Remark 2.22. The maximal grading of a Lie algebra is unique up to equivalence, since maximal split tori are all conjugate (see for instance, [Spr09, Theorem 15.2.6.]). Indeed, by Lemma 2.19(i) the conjugacy implies that any two maximal split tori induce equivalent gradings, so also their universal realizations are equivalent.

Proposition 2.23. *Let $\mathcal{W}$ be a maximal grading of $\mathfrak{g}$ and $\mathcal{V}$ a grading of $\mathfrak{g}$. Then every torsion-free realization of $\mathcal{V}$ is a push-forward of $\mathcal{W}$.*

Proof. Let $\mathcal{V}'$ be the realization of $\mathcal{V}$ as a $\mathfrak{t}^*$-grading induced by a split torus $\mathfrak{t}$ given by Lemma 2.20. Let also $\mathfrak{t}' \supset \mathfrak{t}$ be a maximal split torus in $\mathrm{der}(\mathfrak{g})$ with the induced grading $\mathcal{W}'$. By Lemma 2.19.(ii), the grading $\mathcal{V}'$ is a push-forward of $\mathcal{W}'$. In particular, $\mathcal{V}$ is a coarsening of $\mathcal{W}'$.

Since the maximal grading is unique up to equivalence by Remark 2.22, we may assume that $\mathcal{W}$ is the universal realization of $\mathcal{W}'$. Therefore, by Lemma 2.15 every realization of $\mathcal{V}$ is a push-forward grading of $\mathcal{W}$. □

Remark 2.24. It follows from Proposition 2.23 and the discussion in [Koc09, Section 3.6] that maximal gradings are universal realizations of fine gradings. In [Cor16, Definition 3.18], maximal gradings are defined as the gradings induced by maximal split tori in the automorphism group $\mathrm{Aut}(\mathfrak{g})$. [Cor16, Proposition 3.20] states that maximal gradings in the sense of [Cor16] have a universal property equivalent to Proposition 2.23, so by Lemma 2.11 any such grading is maximal also in the sense of Definition 2.21. The maximal gradings considered in [Fav73] are the gradings induced by maximal split tori.

2.5. **Enumeration of torsion-free gradings.** Following the method suggested in [Koc09, Section 3.7], we now give a simple way to enumerate a complete (and finite) set of universal realizations of gradings of a Lie algebra using the maximal grading. This reduces the proof of Theorem 1.4 to the construction of a maximal grading, which we cover in Subsection 3.3.

For the rest of this section, let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{W} : \mathfrak{g} = \bigoplus_{n\in\mathbb{Z}^k} W_n$ be a maximal grading of $\mathfrak{g}$ with weights Ω. Denote by $\Omega - \Omega$ the difference set $\Omega - \Omega = \{n - m \mid n, m \in \Omega\}$. For a subset $I \subset \Omega - \Omega$, let

$$\pi_I : \mathbb{Z}^k \to \mathbb{Z}^k / \langle I \rangle$$

be the canonical projection. We define the finite set

$$\Gamma = \{(\pi_I)_* \mathcal{W} \mid I \subset \Omega - \Omega,\ \mathbb{Z}^k/\langle I\rangle \text{ is torsion-free}\}.$$

Proposition 2.25. *The set Γ is, up to equivalence, a complete set of universal realizations of torsion-free gradings of $\mathfrak{g}$.*

Proof. Let $\mathcal{V}$ be the universal realization of some torsion-free grading. Due to Lemma 2.14, the grading group of $\mathcal{V}$ is some $\mathbb{Z}^m$. By Proposition 2.23, there exists a homomorphism $f : \mathbb{Z}^k \to \mathbb{Z}^m$ and an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{V} = f_* \Phi(\mathcal{W})$. Let

$$I = \ker(f) \cap (\Omega - \Omega).$$

We are going to show that $\mathcal{V}' = (\pi_I)_*(\mathcal{W})$ is equivalent to $\mathcal{V}$. Then, a posteriori, $\mathbb{Z}^k/\langle I\rangle$ is torsion-free and we have $\mathcal{V}' \in \Gamma$, proving the claim.

First, since $\ker(\pi_I) = \langle I \rangle \subseteq \ker(f)$, by the universal property of quotients there exists a unique homomorphism $\phi : \mathbb{Z}^k/\langle I\rangle \to \mathbb{Z}^m$ such that $f = \phi \circ \pi_I$. In particular,

$$\mathcal{V} = f_*\Phi(\mathcal{W}) = \phi_*(\pi_I)_*\Phi(\mathcal{W}) = \phi_*\Phi(\mathcal{V}'),$$

so $\mathcal{V}$ is a push-forward grading of $\mathcal{V}'$.

Secondly, since also $\ker(f) \cap (\Omega - \Omega) = I \subseteq \ker(\pi_I) \cap (\Omega - \Omega)$, we deduce that $\mathcal{V}$ and $\Phi(\mathcal{V}')$ are realizations of the same grading. Since $\mathcal{V}$ is a universal realization, it follows that $\Phi(\mathcal{V}')$ is a push-forward grading of $\mathcal{V}$. Consequently, $\mathcal{V}'$ is a push-forward grading of $\mathcal{V}$. Since the grading group of a universal realization is generated by the weights, we get that the gradings $\mathcal{V}$ and $\mathcal{V}'$ are equivalent by Lemma 2.11, as wanted. □

Notice that some of the $\mathbb{Z}^k/\langle I\rangle$-gradings in Γ are typically equivalent to each other. From the classification point of view, a more challenging task is to determine the equivalence classes once the set Γ is obtained. In low dimensions, naive methods are enough to separate non-equivalent gradings, and for equivalent ones the connecting automorphism can be found rather easily.

In [HKMT20] we give a representative from each equivalence class of Γ for every 6-dimensional nilpotent Lie algebra over $\mathbb{C}$ and for an extensive class of 7-dimensional Lie algebras over $\mathbb{C}$. The results and the methods for distinguishing the equivalence classes of the obtained gradings are described in more detail in Subsection 4.2.

3. Constructions

3.1. **Stratifications.**

Definition 3.1. A *stratification* (a.k.a. *Carnot grading*) is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} V_n$ such that V_1 generates $\mathfrak{g}$ as a Lie algebra. A Lie algebra $\mathfrak{g}$ is *stratifiable* if it admits a stratification.

In this section we show that constructing a stratification for a Lie algebra (or determining that one does not exist) is a linear problem and, consequently, prove Theorem 1.2. Our method is based on [Cor16, Lemma 3.10], which gives the following characterization of stratifiable Lie algebras:

Lemma 3.2. *A nilpotent Lie algebra $\mathfrak{g}$ is stratifiable if and only if there exists a derivation $\delta\colon \mathfrak{g} \to \mathfrak{g}$ such that the induced map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the identity map. Moreover, a stratification is given by the layers $V_i = \ker(\delta - i)$.*

The condition of Lemma 3.2 is straightforward to check in a basis adapted to the lower central series.

Definition 3.3. The lower central series of a Lie algebra $\mathfrak{g}$ is the decreasing sequence of subspaces

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \mathfrak{g}^{(3)} \supset \cdots,$$

where $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$. A basis $X_1, \ldots, X_n$ of a Lie algebra $\mathfrak{g}$ is *adapted to the lower central series* if for every non-zero $\mathfrak{g}^{(i)}$ there exists an index $n_i \in \mathbb{N}$ such that $X_{n_i}, \ldots, X_n$ is a basis of $\mathfrak{g}^{(i)}$. The *degree* of the basis element X_i is the integer $w_i = \max\{j \in \mathbb{N} : X_i \in \mathfrak{g}^{(j)}\}$.

Proposition 3.4. *Let $X_1, \ldots, X_n$ be a basis adapted to the lower central series of a nilpotent Lie algebra $\mathfrak{g}$ defined over a field F. Let $w_1, \ldots, w_n$ be the degrees of the basis elements and let $c_{ij}^k \in F$ be the structure coefficients in the basis. A linear map $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation that restricts to the identity on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ if and only if*

$$\delta(X_i) = w_i X_i + \sum_{w_j > w_i} a_{ij} X_j \tag{2}$$

such that, for each triple of indices i, j, k such that $w_k > w_i + w_j$, the coefficients $a_{ij} \in F$ satisfy the linear equation

$$c_{ij}^k(w_k - w_i - w_j) = \sum_{w_i < w_h \leq w_k - w_j} a_{ih}c_{hj}^k + \sum_{w_j < w_h \leq w_k - w_i} a_{jh}c_{ih}^k - \sum_{w_i + w_j \leq w_h < w_k} c_{ij}^h a_{hk}. \tag{3}$$

Proof. If $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation that restricts to the identity on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, then by Lemma 3.2 $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$$

such that $\delta|_{V_i} = i \cdot \mathrm{id}$. Since the terms of the lower central series are given in terms of the stratification as $\mathfrak{g}^{(i)} = V_i \oplus \cdots \oplus V_s$, it follows that $\delta(Y) \in i \cdot Y + \mathfrak{g}^{(i+1)}$ for any $Y \in \mathfrak{g}^{(i)}$. That is, a derivation δ restricting to the identity on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is of the form (2) for some coefficients $a_{ij} \in F$.

It is then enough to show that (3) is equivalent to the Leibniz rule

$$\delta([X_i, X_j]) = [\delta(X_i), X_j] + [X_i, \delta(X_j)], \quad \forall i, j \in \{1, \ldots, n\}.$$

Indeed, this would prove that a linear map defined by (2) is a derivation if and only if the coefficients a_{ij} satisfy the linear system (3).

Since the basis X_i is adapted to the lower central series, only the structure coefficients with large enough degrees are non-zero, i.e., we have

$$[X_i, X_j] = \sum_{w_k \geq w_i + w_j} c_{ij}^k X_k. \tag{4}$$

By direct computation using (2) and (4) we get the expressions

$$\begin{aligned}
[\delta(X_i), X_j] &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_i X_k + \sum_{w_h > w_i} \sum_{w_k \geq w_h + w_j} a_{ih} c_{hj}^k X_k \\
[X_i, \delta(X_j)] &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_j X_k + \sum_{w_h > w_j} \sum_{w_k \geq w_i + w_h} a_{jh} c_{ih}^k X_k \\
\delta([X_i, X_j]) &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_k X_k + \sum_{w_h \geq w_i + w_j} \sum_{w_k > w_h} c_{ij}^h a_{hk} X_k
\end{aligned}$$

Denoting $\sum_k B_{ij}^k X_k = \delta([X_i, X_j]) - [\delta(X_i), X_j] - [X_i, \delta(X_j)]$, we find that the equation $B_{ij}^k = 0$ is up to reorganizing terms equivalent to (3).

Finally, we observe that when $w_k \leq w_i + w_j$, the condition $B_{ij}^k = 0$ is automatically satisfied: for $w_k < w_i + w_j$ all of the sums are empty, and for $w_k = w_i + w_j$, the only remaining terms from the sums cancel out as

$$B_{ij}^k = c_{ij}^k w_k - c_{ij}^k w_i - c_{ij}^k w_j = 0.$$

□

The concrete criterion of Proposition 3.4 provides the algorithm of Theorem 1.2.

Algorithm 3.5 (Stratification)**.** *Input: A nilpotent Lie algebra $\mathfrak{g}$. Output: A stratification of $\mathfrak{g}$ or the non-existence of one.*

(1) Construct a basis $X_1, \dots, X_n$ adapted to the lower central series.
(2) Find a derivation δ as in (2) solving the linear system (3). If the system has no solutions, then $\mathfrak{g}$ is not stratifiable.
(3) Return the stratification with the layers $V_i = \ker(\delta - i)$.

3.2. **Positive gradings.**

Definition 3.6. An $\mathbb{R}$-grading $\mathcal{V}\colon \mathfrak{g} = \bigoplus_{\alpha\in\mathbb{R}} V_\alpha$ is *positive* if $\alpha > 0$ for all the weights of $\mathcal{V}$. If such a grading exists for $\mathfrak{g}$, then $\mathfrak{g}$ is said to be *positively gradable.*

One of our main goals is to determine when a grading admits a positive realization, i.e., can be realized as a positive grading. A characterization is given in [Cor16, Proposition 3.22]. In the lemma and proposition below, we provide constructive proofs for this characterization.

Lemma 3.7. *Let $m \geq 1$ and let $\mathcal{V}$ be a $\mathbb{Z}^m$-grading of a Lie algebra $\mathfrak{g}$. Suppose the convex hull of the set of weights of $\mathcal{V}$ does not contain the origin. Then there exists a homomorphism $f\colon \mathbb{Z}^m \to \mathbb{Z}$ whose restriction on the weights is injective and positive.*

Proof. Let us consider the natural embedding of $\mathbb{Z}^m$ into $\mathbb{Q}^m$. Using the canonical inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{Q}^m$, we define for each vector $v \in \mathbb{Q}^m$, the corresponding open half-space

$$M_v = \{x \in \mathbb{Q}^m : \langle v, x\rangle > 0\}.$$

Denote by $\Omega = \{\alpha_1, \dots, \alpha_N\}$ the set of weights of $\mathcal{V}$. Recall that the convex hull of a set is the intersection of all the affine half-spaces containing the set. Hence, because the convex hull of Ω does not contain the origin, it is contained in an open half-space $M_{v_0} \subset \mathbb{Q}^m$. Moreover, there exists a neighborhood B of v_0 such that the convex hull of Ω is contained in every half-space M_v with $v \in B$.

By construction all the inner products $\langle v, \alpha_i\rangle$ with $\alpha_i \in \Omega$ and $v \in B$ are strictly positive. Since B has non-empty interior, we may choose some $v \in B$ such that all the numbers $\langle v, \alpha_1\rangle, \dots, \langle v, \alpha_N\rangle$ are strictly positive and distinct. Rescaling v to eliminate denominators, we obtain a vector $\tilde{v} \in \mathbb{Z}^m$, and the map $f(\cdot) = \langle \tilde{v}, \cdot\rangle$ is the required homomorphism $\mathbb{Z}^m \to \mathbb{Z}$. Concretely, a valid vector $\tilde{v}$ can be found directly by just enumerating the points of $\mathbb{Z}^k$ with increasing distance from the origin and testing one by one if all the inner products with the weights are positive and distinct. □

Proposition 3.8. *Let $\mathcal{W}$ be a torsion-free grading. Then $\mathcal{W}$ admits a positive realization if and only if the convex hull of the set of weights of the universal realization of $\mathcal{W}$ does not contain the origin.*

Proof. We only need to prove the forward implication due to Lemma 3.7. Let $\mathcal{V}$ be a positive realization of $\mathcal{W}$ and let $\widetilde{\mathcal{W}}$ be the universal realization of $\mathcal{W}$, which by Lemma 2.14 is a $\mathbb{Z}^k$-grading. Then by the definition of a universal realization there is a homomorphism $f\colon \mathbb{Z}^k \to \mathbb{R}$ such that $\mathcal{V} = f_*\widetilde{\mathcal{W}}$. Consider the vector $v = (f(e_1), \ldots, f(e_k)) \in \mathbb{R}^k$, where $e_1, \ldots, e_k$ are the standard basis vectors of $\mathbb{Z}^k$, and express f as $f(\cdot) = \langle v, \cdot \rangle$. Since $\mathcal{V}$ is a positive grading, then for all weights α of $\widetilde{\mathcal{W}}$ we have $f(\alpha) > 0$, that is, $\langle v, \alpha \rangle > 0$. Hence all the weights belong to the open half-space determined by the vector v, and so the origin is not contained in their convex hull. □

The above results give the following algorithm for Theorem 1.5.(i). We stress that we do not need to assume that the base field of $\mathfrak{g}$ is algebraically closed.

Algorithm 3.9 (Positive realization)**.** *Input: A torsion-free grading $\mathcal{V}$ for a Lie algebra $\mathfrak{g}$. Output: A positive realization of $\mathcal{V}$ or the non-existence of one.*

1. *Compute the universal realization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ using Algorithm 2.13. Let $\mathbb{Z}^k$ be the grading group of $\widetilde{\mathcal{V}}$.*
2. *If the convex hull of the weights of $\widetilde{\mathcal{V}}$ contains the origin, then no positive realization exists.*
3. *Otherwise, find a vector $v \in \mathbb{Z}^k$ so that the homomorphism $f\colon \mathbb{Z}^k \to \mathbb{Z}$, $f(\cdot) = \langle v, \cdot \rangle$ maps all weights of $\widetilde{\mathcal{V}}$ to distinct positive integers.*
4. *Return the push-forward grading $f_*\widetilde{\mathcal{V}}$.*

The algorithm for Theorem 1.5.(ii) is somewhat similar to Algorithm 3.9. Suppose we have an S-grading $\mathcal{V}$ for a finitely generated abelian group S, and we want to find a homomorphism $S \to \mathbb{R}$ turning it into a positive grading. If some element of S has torsion, then no such homomorphism can exist. Otherwise, S is isomorphic to $\mathbb{Z}^k$ for some $k \geq 1$ and we may follow steps 2-3 of the above algorithm to obtain a homomorphism $S \to \mathbb{R}$ giving a positive realization if one exists.

Remark 3.10. The argument of Lemma 3.7 can also be used to find a realization over $\mathbb{Z}$ for any torsion-free grading $\mathcal{V}$. Indeed, first consider the universal realization of $\mathcal{V}$ over some $\mathbb{Z}^k$. Then disregarding the discussion about half-spaces and positivity, find a push-forward to $\mathbb{Z}$ by constructing a vector v for which the projection is injective on the weights.

Remark 3.11. The results we have established can also be used to explicitly enumerate the positive gradings of a given Lie algebra $\mathfrak{g}$ over an algebraically closed field, in two different senses.

(i) Consider the maximal grading $\mathcal{V}$ of $\mathfrak{g}$ over $\mathbb{Z}^k$. Up to automorphism, positive gradings of $\mathfrak{g}$ are given by the projections from $\mathbb{Z}^k$ to $\mathbb{R}$ mapping the weights of $\mathcal{V}$ to strictly positive numbers. A parametrisation of these projections gives a parametrisation of positive gradings.
(ii) Construct all the gradings of $\mathfrak{g}$ as in Proposition 2.25 (using a maximal grading constructed in Algorithm 3.12). Then check one by one which of them admit positive realizations. This produces a finite list of positive gradings so that every positive grading of $\mathfrak{g}$ is equivalent in the sense of Definition 2.9 to one of the elements on the list.

The algorithm of Theorem 1.3 is given by applying Algorithm 3.9 to the maximal grading of a Lie algebra. Indeed, if some grading admits a positive realization, then by Proposition 2.23 the maximal grading admits a positive realization as well. For the maximal grading, a positive realization if one exists is given by Algorithm 3.9.

The existence of a positive realization of a grading can also be phrased as the existence of a solution to a linear system. This viewpoint gives rise to an alternate elementary algorithm to determine whether a positive realization exists, as we explain in Appendix A.

3.3. **Maximal gradings.** In this section we prove Theorem 1.1 by providing an algorithm to construct a maximal grading for a Lie algebra $\mathfrak{g}$ defined over an algebraically closed field of characteristic zero. In this setting, every torus is split. The method we use to compute maximal gradings is the following.

Algorithm 3.12 (Maximal grading)**.** *Input: A Lie algebra $\mathfrak{g}$ over an algebraically closed field F. Output: A maximal grading of $\mathfrak{g}$.*

(1) Compute a basis for the derivation algebra $\mathrm{der}(\mathfrak{g})$*. Set $B = \emptyset$.*
(2) Determine the $\mathfrak{t}^*$-grading $\mathcal{V} : \mathfrak{g} = \bigoplus_\lambda V_\lambda$ induced by the torus $\mathfrak{t} = \langle B \rangle$.
(3) Compute a basis $A_1, \ldots, A_n$ for the centralizer $C(\mathfrak{t}) \subset \mathrm{der}(\mathfrak{g})$.
(4) Compute the adjoint representation $\mathrm{ad}\colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$*.*
(5) Compute a basis $K_1, \ldots, K_m$ for $\ker(\mathrm{ad}) \subset C(\mathfrak{t})$*. If $K_i \notin \mathfrak{t}$ for some $i = 1, \ldots, m$, extend $\mathfrak{t}$ by K_i and go back to step 2.*
(6) Repeat for each $A = A_i$ and $A = A_i + A_j$, $i, j = 1, \ldots, n$: compute the Jordan decomposition $A = A_s + A_n$. If $A_s \notin \mathfrak{t}$, extend $\mathfrak{t}$ by A_s and go back to step 2.
(7) Compute and return the universal realization of the grading $\mathcal{V}$.

The rest of the section is devoted to proving the correctness of Algorithm 3.12 and to explaining the steps in more detail. Step 6 is the most involved part.

Step 1 is straightforward linear algebra. In step 2, the grading induced by the torus $\mathfrak{t}$ has a concrete description in terms of a fixed basis of $\mathfrak{t}$. Namely, a basis $\delta_1, \ldots, \delta_k$ defines an isomorphism $\mathfrak{t}^* \to F^k$ and hence an equivalent push-forward grading over F^k. Expanding out the construction of Lemma 2.17 shows that the push-forward grading has the layers

$$V_\lambda = V_{(\lambda_1,\ldots,\lambda_k)} = \bigcap_{i=1}^{k} E_{\delta_i}^{\lambda_i},$$

where $E_{\delta_i}^{\lambda_i}$ is the (possibly zero) eigenspace for the eigenvalue λ_i of the derivation δ_i.

Step 3 is another straightforward linear algebra computation. In step 4, the key observation is that any linear map $A \in C(\mathfrak{t})$ preserves the eigenspaces of all the derivations $\delta \in \mathfrak{t}$. Hence such a linear map A also preserves the layers V_λ of the F^k-grading. It follows that each map $\mathrm{ad}(A)$ restricts to a linear map $\mathrm{ad}(A) \colon \mathfrak{gl}(V_\lambda) \to \mathfrak{gl}(V_\lambda)$ for each weight λ. The direct sum of these representations gives the representation $\mathrm{ad} \colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$.

Step 5 captures the situation when the torus $\mathfrak{t}$ can be extended without refining the grading. Indeed, the elements of the kernel of ad are the elements $A \in C(\mathfrak{t})$ whose restrictions commute with all other maps in $\mathfrak{gl}(V_\lambda)$ for each weight λ. That is, they are the maps $A \in C(\mathfrak{t})$ such that each $A|_{V_\lambda}$ is a multiple of the identity. The eigenspaces of such maps are sums of the layers V_λ, so they do not further refine the grading induced by $\mathfrak{t}$, as seen in the following example.

Example 3.13. Let $\mathfrak{h}$ be the Heisenberg Lie algebra with the only bracket $[X, Y] = Z$. Consider the derivation

$$\delta \colon \mathfrak{h} \to \mathfrak{h}, \quad \delta(X) = X, \quad \delta(Y) = 2Y, \quad \delta(Z) = 3Z.$$

The grading induced by δ is $\mathfrak{h} = V_1 \oplus V_2 \oplus V_3 = \langle X \rangle \oplus \langle Y \rangle \oplus \langle Z \rangle$.

The centralizer of δ in $\mathrm{der}(\mathfrak{h})$ is the two-dimensional space $C(\delta) = \langle \delta_1, \delta_2 \rangle$, where the two basis derivations are defined by

$$\begin{aligned} &\delta_1(X) = X, \qquad &&\delta_1(Y) = 0, \qquad &&\delta_1(Z) = Z, \\ &\delta_2(X) = 0, \qquad &&\delta_2(Y) = Y, \qquad &&\delta_1(Z) = Z. \end{aligned}$$

The one-dimensional Lie algebras $\mathfrak{gl}(V_i)$ are all abelian, so the adjoint representation $\mathrm{ad} \colon C(\delta) \to \mathfrak{gl}(\mathfrak{gl}(V_1)) \oplus \mathfrak{gl}(\mathfrak{gl}(V_2)) \oplus \mathfrak{gl}(\mathfrak{gl}(V_3))$ is just the zero map. Both $\{\delta, \delta_1\}$ and $\{\delta, \delta_2\}$ span strictly bigger tori than $\{\delta\}$, but neither torus further refines the original grading $\mathfrak{h} = V_1 \oplus V_2 \oplus V_3$: for instance, the grading induced by $\langle \delta, \delta_1 \rangle$ is $V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(3,1)} = \langle X \rangle \oplus \langle Y \rangle \oplus \langle Z \rangle$.

Step 6 is the most intricate part of Algorithm 3.12. To prove its correctness, we need to show that if $A_s \in \mathfrak{t}$ for all basis elements $A = A_i$ and all their sums $A = A_i + A_j$, then the torus $\mathfrak{t}$ is maximal. The proof is based on the efficient criterion of [dG17, Proposition 2.6.11]:

Lemma 3.14. *Let* $\mathfrak{c}$ *be a Lie algebra and let* $X_1, \dots, X_n$ *be a basis of* $\mathfrak{c}$*. If* $\mathrm{ad}(X_i)$ *is nilpotent for* $1 \leq i \leq n$ *and* $\mathrm{ad}(X_i + X_j)$ *is nilpotent for all* $1 \leq i < j \leq n$*, then* $\mathrm{ad}(X)$ *is nilpotent for all* $X \in \mathfrak{c}$*.*

To make use of the criterion Lemma 3.14, we also need the fact the Jordan decomposition is preserved by the adjoint representation.

Lemma 3.15. *Let* F *be a field of characteristic zero. Let* $A \in \mathfrak{gl}(n, F)$ *be any linear map and* $A = A_s + A_n$ *its Jordan decomposition. Then* $\mathrm{ad}(A) = \mathrm{ad}(A_s) + \mathrm{ad}(A_n)$ *is the Jordan decomposition of the map* $\mathrm{ad}(A) \colon \mathfrak{gl}(n, F) \to \mathfrak{gl}(n, F)$*.*

Proof. By [dG17, Proposition 2.2.5], since the field F is perfect (as a field of characteristic zero), the adjoint map preserves both semisimplicity and nilpotency, so the map $\mathrm{ad}(A_s)$ is semisimple and the map $\mathrm{ad}(A_n)$ is nilpotent. Moreover, since the maps A_s and A_n commute, the Jacobi identity implies that the maps $\mathrm{ad}(A_s)$ and $\mathrm{ad}(A_n)$ also commute. The claim follows from the uniqueness of the Jordan decomposition. □

With the above results, we are able to conclude that if the semisimple parts of A_i and $A_i + A_j$ are contained in $\mathfrak{t}$ for all basis elements A_i, then the torus $\mathfrak{t}$ in step 6 of Algorithm 3.12 is maximal. First, by step 5 we have $\mathfrak{t} = \ker(\mathrm{ad})$ for the restricted adjoint representation $\mathrm{ad} \colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$ defined in step 4. Then for any $A \in C(\mathfrak{t})$ by Lemma 3.15 we find that $A_s \in \mathfrak{t}$ if and only if $\mathrm{ad}(A) = \mathrm{ad}(A_n)$, that is, if and only if $\mathrm{ad}(A)$ is nilpotent. By Lemma 3.14, if all $\mathrm{ad}(A_i)$ and $\mathrm{ad}(A_i + A_j)$ are nilpotent, then $\mathrm{ad}(A)$ is nilpotent for all $A \in C(\mathfrak{t})$. Hence $A_s \in \mathfrak{t}$ for all $A \in C(\mathfrak{t})$. In other words, no semisimple element $A_s \in C(\mathfrak{t}) \setminus \mathfrak{t}$ exists, so $\mathfrak{t}$ is maximal.

The final part of Algorithm 3.12 is step 7, where we replace the indexing by eigenvalues of the derivations of $\mathfrak{t}$ with indexing over some $\mathbb{Z}^k$ given by the universal realization. The precise method was described earlier in Algorithm 2.13. Since the construction of the first six steps of Algorithm 3.12 leads to a maximal torus of $\mathrm{der}(\mathfrak{g})$, by Definition 2.21 the output is a maximal grading of $\mathfrak{g}$.

Remark 3.16. The only part where we use the assumption that the base field is algebraically closed is in step 6. The significance of the assumption is that the Jordan decomposition and [dG17, Proposition 2.6.11] give us an efficient method to construct semisimple elements in $C(\mathfrak{t}) \setminus \mathfrak{t}$.

If the base field is not algebraically closed, we need to explicitly require that the constructed elements of $C(\mathfrak{t}) \setminus \mathfrak{t}$ are diagonalizable. The subset of diagonalizable elements of $C(\mathfrak{t}) \setminus \mathfrak{t}$ is a semialgebraic set, and constructions to extract points from such sets exist, see for instance [BPR06, Section 13] on the existential theory of the reals. The problem is that these methods are practical only in low dimensions, and the construction would be needed in dimension $\dim \mathfrak{gl}(\mathfrak{g}) = \dim(\mathfrak{g})^2$.

For Lie algebras defined over finite fields, more efficient randomized algorithms to find split tori exist, see [CM09] and [Roo13].

4. Applications

4.1. **Structure from maximal gradings.** In this subsection we show how maximal gradings may be used to find some structural information of Lie algebras. We start by studying how maximal gradings reveal the structure of a direct product. A similar result can be found in 1.6.5. of [Fav73].

Example 4.1. Consider the Lie algebra $L_{6,22}(1)$ in [CdGS12] with basis $\{X_1, \dots, X_6\}$, where the only non-zero bracket relations are

$$[X_1, X_2] = X_5, \quad [X_1, X_3] = X_6, \quad [X_2, X_4] = X_6, \quad [X_3, X_4] = X_5.$$

In a basis $\{Y_1, \dots, Y_6\}$ adapted to the maximal grading, the bracket relations are

$$[Y_1, Y_2] = Y_3, \quad [Y_4, Y_5] = Y_6.$$

From these bracket relations one sees more easily that the Lie algebra $L_{6,22}(1)$ is isomorphic to $L_{3,2} \times L_{3,2}$, where $L_{3,2}$ is the first Heisenberg Lie algebra.

We say that a split torus $\mathfrak{t} \subset \operatorname{der}(\mathfrak{g})$ is *non-degenerate* if the intersection of the kernels of the maps $D \in \mathfrak{t}$ is trivial. That is, a split torus is non-degenerate if and only if the $\mathfrak{t}^*$-grading it induces does not have zero as a weight.

We expect that the following result is known even without the non-degeneracy assumption, however we have been unable to locate a reference. We will therefore give a direct proof of the simpler claim.

Lemma 4.2. *Let $\mathfrak{t}_1 \subset \operatorname{der}(\mathfrak{g}_1)$ and $\mathfrak{t}_2 \subset \operatorname{der}(\mathfrak{g}_2)$ be non-degenerate maximal split tori. Then $\mathfrak{t}_1 \times \mathfrak{t}_2$ is a maximal split torus in $\operatorname{der}(\mathfrak{g}_1 \times \mathfrak{g}_2)$.*

Proof. Denoting $\mathfrak{t} = \mathfrak{t}_1 \times \mathfrak{t}_2$, let $D \in C(\mathfrak{t})$ be a diagonalizable derivation in the centralizer $C(\mathfrak{t})$. To show the maximality of $\mathfrak{t}$, it suffices to show that $D \in \mathfrak{t}$. In a basis adapted to the product we may represent

$$D = \begin{bmatrix} E_1 & F_1 \\ F_2 & E_2 \end{bmatrix},$$

where $E_1 \in \operatorname{der}(\mathfrak{g}_1)$, $E_2 \in \operatorname{der}(\mathfrak{g}_2)$, and $F_1 \colon \mathfrak{g}_2 \to \mathfrak{g}_1$ and $F_2 \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ are some linear maps. We are going to demonstrate that $E_1 \in \mathfrak{t}_1$, $E_2 \in \mathfrak{t}_2$ and $F_1 = F_2 = 0$, which would prove that $D = E_1 \times E_2 \in \mathfrak{t}$.

Let $D_1 \in \mathfrak{t}_1$. By assumption D commutes with $D_1 \times 0 \in \mathfrak{t}$, so a simple computation shows that E_1 commutes with D_1 and $D_1 F_1 = 0$. Since D_1 is arbitrary, we obtain $E_1 \in C(\mathfrak{t}_1)$. From the fact that $D_1 F_1 = 0$ for every $D_1 \in \mathfrak{t}_1$ we get

$$\operatorname{Im}(F_1) \subset \bigcap_{D_1 \in \mathfrak{t}_1} \ker(D_1) = \{0\},$$

where the last equality follows from the non-degeneracy of $\mathfrak{t}_1$. Consequently, $F_1 = 0$.

A similar argument shows that $E_2 \in C(\mathfrak{t}_2)$ and $F_2 = 0$. Since D is assumed diagonalizable, it follows that E_1 and E_2 are diagonalizable. Then by maximality of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ we have $E_1 \in \mathfrak{t}_1$ and $E_2 \in \mathfrak{t}_2$, which shows that $D = E_1 \times E_2 \in \mathfrak{t}$. □

For gradings, the above lemma implies the following. Suppose $\mathcal{V} : \mathfrak{g}_1 = \bigoplus_{\alpha\in A} V_\alpha$ and $\mathcal{W} : \mathfrak{g}_2 = \bigoplus_{\beta\in B} W_\beta$ are maximal gradings of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and suppose zero is not a weight for either $\mathcal{V}$ or $\mathcal{W}$. Then

$$\mathcal{V} \times \mathcal{W} : \Big(\bigoplus_{(\alpha,0)\in A\times B} V_\alpha \times \{0\} \Big) \oplus \Big(\bigoplus_{(0,\beta)\in A\times B} \{0\} \times W_\beta \Big) \tag{5}$$

is a maximal grading of $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Indeed, the gradings $\mathcal{V}$ and $\mathcal{W}$ are the universal realizations of gradings induced by the respective maximal split tori $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. By Lemma 4.2, the product torus $\mathfrak{t}_1 \times \mathfrak{t}_2$ is maximal. The universal realization of the grading induced by $\mathfrak{t}_1 \times \mathfrak{t}_2$ is equivalent to the product grading (5).

For a grading $\mathfrak{g} = \bigoplus_{\alpha\in\Omega} V_\alpha$, consider the graph with vertices Ω defined as follows: Whenever $0 \neq [V_\alpha, V_\beta] \subset V_\gamma$, we define edges between all the three vertices $\alpha, \beta, \gamma \in \Omega$. If the graph Ω admits a partition $\Omega = \Omega_1 \sqcup \Omega_2$ such that no edges exist between Ω_1 and Ω_2, then the Lie algebra $\mathfrak{g}$ is a direct product of the ideals $\mathfrak{g}_1 = \bigoplus_{\alpha\in\Omega_1} V_\alpha$ and $\mathfrak{g}_2 = \bigoplus_{\beta\in\Omega_2} V_\beta$. In this situation we say the grading $\mathcal{V}$ *detects the product structure* $\mathfrak{g}_1 \times \mathfrak{g}_2$ of the Lie algebra $\mathfrak{g}$. We gather the observations made above into the following proposition.

Proposition 4.3. *If a Lie algebra $\mathfrak{g}$ is decomposable and the maximal gradings of the factor Lie algebras do not have zero as a weight, then the maximal grading of $\mathfrak{g}$ detects the product structure.*

We remark that while maximal gradings are able to detect product structures as indicated above, they are not able to detect some other algebraic properties. The Lie algebra $L_{6,24}(1)$ in [CdGS12] provides examples of two such phenomena. First, the layers of its maximal grading are not contained in the terms of its lower central series (this behavior can be also achieved by examples where the maximal grading is very coarse). Secondly, this Lie algebra has a "nice" basis (see [CR19] for the precise definition and its motivation), but it can be shown that no basis adapted to a maximal grading is nice.

Despite these negative results, maximal gradings have another structural application in simplifying the problem of deciding whether two Lie algebras are isomorphic or not.

Remark 4.4. If two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, then any isomorphism maps the maximal grading of $\mathfrak{g}_1$ to a maximal grading of $\mathfrak{g}_2$. Therefore, if the maximal gradings of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are given, then deciding if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic reduces to determining the existence of an isomorphism between the maximal gradings. In many cases this is significantly easier than naively solving the original isomorphism problem. For example, in low dimensions, the majority of the layers of the maximal grading are one-dimensional, in which case searching for possible isomorphisms becomes a combinatorial problem.

4.2. **Classification of gradings in low dimension.** Following the strategy outlined in [Koc09, Section 3.7], we classify torsion-free gradings, i.e., gradings that admit a torsion-free realization, in nilpotent Lie algebras of dimension up to 6 over $\mathbb{C}$. The main part of the classification is the construction of a maximal grading (Algorithm 3.12) and the enumeration of torsion-free gradings (Proposition 2.25). Here we will give a brief overview of the gradings of each Lie algebra.

We give a complete listing of the universal realizations for the 669 equivalence classes of gradings for the 46 complex Lie algebras of dimension up to 6 in [HKMT20]. We also include a similar listing for an extensive (but incomplete) family of 7 dimensional Lie algebras over $\mathbb{C}$. The listing in dimension 7 is incomplete because there are a few uncountable families of 7 dimensional complex Lie algebras that depend on a complex parameter λ. In these cases, following the study carried out in [Mag08], we focus on those singular values of λ for which either the Lie algebra cohomology or the adjoint cohomology have different dimensions compared to the rest of the Lie algebras in the same family. We will also include a few examples corresponding to non-singular values.

As a starting point we used the classifications of nilpotent Lie algebras given in [dG07] for dimensions less than 6, [CdGS12] for dimension 6, and [Gon98] for dimension 7. The classification up to dimension 6 has a pre-existing computer implementation in the GAP package [CdGSGT18]. However these Lie algebras are not always given in a basis adapted to any maximal grading, so we first compute the maximal grading using the methods described in Subsection 3.3 and switch to a basis adapted to the resulting grading.

The presentations we use for the nilpotent Lie algebras up to dimension 6 are listed in Table 1. The Lie brackets $[Y_a, Y_b] = Y_c$ are listed in the condensed form $ab = c$. Lie algebras $\mathfrak{g} \times \mathbb{R}^d$ with abelian factors have identical structure coefficients with the nonabelian factor $\mathfrak{g}$ and are omitted from the list. For example $L_{4,2} = L_{3,2} \times \mathbb{R}$ has the basis $Y_1, \dots, Y_4$ with the bracket relation $[Y_1, Y_2] = Y_3$ from $L_{3,2}$.

With all the maximal gradings computed, we enumerate all torsion-free gradings as in Proposition 2.25. For the classification up to equivalence, we first introduce some easy-to-check invariants for gradings. Recall that by Lemma 2.14, the grading groups of the obtained gradings are isomorphic to some groups $\mathbb{Z}^k$. The dimension k is called the

$L_{3,2}$	$12 = 3$					
$L_{4,3}$	$12 = 3$	$13 = 4$				
$L_{5,4}$	$41 = 5$	$23 = 5$				
$L_{5,5}$	$13 = 4$	$14 = 5$	$32 = 5$			
$L_{5,6}$	$12 = 3$	$13 = 4$	$14 = 5$	$23 = 5$		
$L_{5,7}$	$12 = 3$	$13 = 4$	$14 = 5$			
$L_{5,8}$	$12 = 3$	$14 = 5$				
$L_{5,9}$	$12 = 3$	$23 = 4$	$13 = 5$			
$L_{6,10}$	$23 = 4$	$51 = 6$	$24 = 6$			
$L_{6,11}$	$12 = 3$	$13 = 5$	$15 = 6$	$23 = 6$	$24 = 6$	
$L_{6,12}$	$23 = 4$	$24 = 5$	$31 = 6$	$25 = 6$		
$L_{6,13}$	$13 = 4$	$14 = 5$	$32 = 5$	$15 = 6$	$42 = 6$	
$L_{6,14}$	$12 = 3$	$13 = 4$	$14 = 5$	$23 = 5$	$25 = 6$	$43 = 6$
$L_{6,15}$	$12 = 3$	$13 = 4$	$14 = 5$	$23 = 5$	$15 = 6$	$24 = 6$
$L_{6,16}$	$12 = 3$	$13 = 4$	$14 = 5$	$25 = 6$	$43 = 6$	
$L_{6,17}$	$21 = 3$	$23 = 4$	$24 = 5$	$13 = 6$	$25 = 6$	
$L_{6,18}$	$12 = 3$	$13 = 4$	$14 = 5$	$15 = 6$		
$L_{6,19}(-1)$	$12 = 3$	$14 = 5$	$25 = 6$	$43 = 6$		
$L_{6,20}$	$12 = 3$	$14 = 5$	$15 = 6$	$23 = 6$		
$L_{6,21}(-1)$	$12 = 3$	$23 = 4$	$13 = 5$	$14 = 6$	$25 = 6$	
$L_{6,22}(0)$	$24 = 5$	$41 = 6$	$23 = 6$			
$L_{6,22}(1)$	$12 = 3$	$45 = 6$				
$L_{6,23}$	$12 = 3$	$14 = 5$	$15 = 6$	$42 = 6$		
$L_{6,24}(0)$	$13 = 4$	$34 = 5$	$14 = 6$	$32 = 6$		
$L_{6,24}(1)$	$12 = 3$	$23 = 5$	$24 = 5$	$13 = 6$		
$L_{6,25}$	$12 = 3$	$13 = 4$	$15 = 6$			
$L_{6,26}$	$12 = 3$	$24 = 5$	$14 = 6$			
$L_{6,27}$	$12 = 3$	$13 = 4$	$25 = 6$			
$L_{6,28}$	$12 = 3$	$23 = 4$	$13 = 5$	$15 = 6$		

TABLE 1. Lie algebras of dimension up to 6 over $\mathbb{C}$ in a basis adapted to a maximal grading.

rank of the grading. We recall also an invariant from [Koc09, Section 3.2]: The *type* of a grading is the tuple $(n_1, n_2, \dots, n_k)$, where k is the dimension of the largest layer, and each n_i is the number of i-dimensional layers.

From the full list of torsion-free gradings, we initially collect together gradings using the following criteria:

(1) The ranks of the gradings are equal.
(2) The types of the gradings are equal.
(3) There exists a homomorphism between the grading groups of the universal realizations mapping layers to layers of equal dimensions.

In this way we get for each Lie algebra families $I_1, I_2, \ldots, I_k$ of gradings such that the gradings of I_i and I_j are not equivalent for $i \neq j$.

To compute the precise equivalence classes, we naively check if the gradings within each family I_i are equivalent. For each pair of an A-grading $\mathfrak{g} = \bigoplus_{\alpha\in A} V_\alpha$ and a B-grading $\mathfrak{g} = \bigoplus_{\beta\in B} W_\beta$, there are usually only a few homomorphisms $f\colon A \to B$ with $\dim V_\alpha = \dim W_{f(\beta)}$. For each such homomorphism f, we need to check whether there exists an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\Phi(V_\alpha) = W_{f(\beta)}$ for all weights α. These identities define a system of quadratic equations. Since we are working over an algebraically closed field, the system has no solution if and only if 1 is contained in the ideal defined by the polynomial equations. The dimensions of the layers are generally quite small in the cases we need to check, so Gröbner basis methods work well.

For nilpotent Lie algebras of dimension up to 6, an overview of our classification of gradings is compiled in Table 2. For each Lie algebra, we list its label in the classification of [CdGS12], the rank of its maximal grading (k), whether it is stratifiable or not (s?), the number of gradings (#), and the number of gradings with a positive realization ($\#\mathbb{Z}_+$).

Example 4.5. We present our method of classifying gradings explicitly in the simple case of the Lie algebra $L_{4,2} = L_{3,2} \times \mathbb{R}$ given in the basis $Y_1, \ldots, Y_4$ with the only bracket $[Y_1, Y_2] = Y_3$. The maximal grading is over $\mathbb{Z}^3$ with the layers

$$V_{(1,0,0)} = \langle Y_1\rangle, \quad V_{(0,1,0)} = \langle Y_2\rangle, \quad V_{(1,1,0)} = \langle Y_3\rangle, \quad V_{(0,0,1)} = \langle Y_4\rangle.$$

Ignoring scalar multiples, the difference set $\Omega - \Omega$ of weights consists of the 6 elements e_1, e_2, $e_1 - e_2$, $e_1 - e_3$, $e_2 - e_3$, and $e_1 + e_2 - e_3$, where e_1, e_2, e_3 are the standard basis elements of $\mathbb{Z}^3$. Subsets of these points span the trivial subspace, 6 one-dimensional subspaces, 7 two-dimensional subspaces $\langle e_1, e_2\rangle$, $\langle e_1, e_3\rangle$, $\langle e_2, e_3\rangle$, $\langle e_1 - e_3, e_2\rangle$, $\langle e_1, e_2 - e_3\rangle$, $\langle e_1 - e_3, e_2 - e_3\rangle$, $\langle 2e_1 - e_3, 2e_2 - e_3\rangle$, and the full space $\mathbb{Z}^3$.

In this case, each of these 15 subspaces S defines a torsion-free quotient $\mathbb{Z}^3/S$. For instance parametrizing the quotient $\pi\colon \mathbb{Z}^3 \to \mathbb{Z}^3/\langle e_1 - e_3, e_2 - e_3\rangle$ as $\mathbb{Z}$ using the complementary line $\mathbb{Z}e_3$ gives the weights

$$\pi(e_1) = \pi(e_2) = \pi(e_3) = 1, \quad \pi(e_1 + e_2) = 2,$$

so a push-forward grading for the quotient $\mathbb{Z}^3/\langle e_1 - e_3, e_2 - e_3\rangle$ is the $\mathbb{Z}$-grading

$$V_1 = \langle Y_1, Y_2, Y_4\rangle, \quad V_2 = \langle Y_3\rangle.$$

To determine the distinct equivalence classes out of the 15 gradings, we first consider the simple criteria listed earlier. The trivial grading and the maximal grading are distinguished by the rank. The six $\mathbb{Z}^2$-gradings all have 2 one-dimensional layers and 1 two-dimensional layer. There exists a homomorphism that preserves the dimensions of the layers for two pairs of the gradings: one between the quotients by $\langle e_1\rangle$ and $\langle e_2\rangle$, and one between the quotients by $\langle e_1 - e_3\rangle$ and $\langle e_2 - e_3\rangle$.

Name	k	s?	#	$\#\mathbb{Z}_+$	Name	k	s?	#	$\#\mathbb{Z}_+$
$L_{2,1}$	2	✓	2	2	$L_{6,9}$	3	✓	17	8
$L_{3,1}$	3	✓	3	3	$L_{6,10}$	3		23	8
$L_{3,2}$	2	✓	4	2	$L_{6,11}$	1		2	1
$L_{4,1}$	4	✓	5	5	$L_{6,12}$	2		9	4
$L_{4,2}$	3	✓	11	6	$L_{6,13}$	2		8	3
$L_{4,3}$	2	✓	6	2	$L_{6,14}$	1		2	1
$L_{5,1}$	5	✓	7	7	$L_{6,15}$	1		2	1
$L_{5,2}$	4	✓	26	15	$L_{6,16}$	2	✓	8	2
$L_{5,3}$	3	✓	22	9	$L_{6,17}$	1		2	1
$L_{5,4}$	3	✓	9	4	$L_{6,18}$	2	✓	8	2
$L_{5,5}$	2		7	3	$L_{6,19}(-1)$	3	✓	21	6
$L_{5,6}$	1		2	1	$L_{6,20}$	2	✓	8	3
$L_{5,7}$	2	✓	7	2	$L_{6,21}(-1)$	2	✓	6	2
$L_{5,8}$	3	✓	14	6	$L_{6,22}(0)$	3	✓	18	8
$L_{5,9}$	2	✓	5	2	$L_{6,22}(1)$	4	✓	32	15
$L_{6,1}$	6	✓	11	11	$L_{6,23}$	2		8	4
$L_{6,2}$	5	✓	52	31	$L_{6,24}(0)$	2		8	4
$L_{6,3}$	4	✓	60	27	$L_{6,24}(1)$	2		5	2
$L_{6,4}$	4	✓	29	13	$L_{6,25}$	3	✓	29	11
$L_{6,5}$	3		29	15	$L_{6,26}$	3	✓	10	5
$L_{6,6}$	2		8	6	$L_{6,27}$	3	✓	32	13
$L_{6,7}$	3	✓	31	11	$L_{6,28}$	2	✓	8	3
$L_{6,8}$	4	✓	52	25					

Table 2. Gradings of Lie algebras up to dimension 6 over $\mathbb{C}$

Out of the seven $\mathbb{Z}$-gradings, the four quotients by

$$\langle e_1, e_2\rangle, \langle e_1 - e_3, e_2\rangle, \langle e_1, e_2 - e_3\rangle, \langle e_1 - e_3, e_2 - e_3\rangle$$

define gradings with 1 one-dimensional layer and 1 three-dimensional layer, and the three quotients by

$$\langle e_1, e_3\rangle, \langle e_2, e_3\rangle, \langle 2e_1 - e_3, 2e_2 - e_3\rangle$$

define gradings with 2 two-dimensional layers. In both families there is exactly one pair of gradings admitting a homomorphism: the pair $\langle e_1 - e_3, e_2\rangle$ and $\langle e_1, e_2 - e_3\rangle$, and the pair $\langle e_1, e_3\rangle$ and $\langle e_2, e_3\rangle$.

In all of these cases, the homomorphism between the quotients is induced by the isomorphism $f\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ swapping e_1 and e_2. All of the mentioned pairs of $\mathbb{Z}^2$- and $\mathbb{Z}$-gradings are in fact equivalent, since there is a corresponding Lie algebra automorphism swapping the basis elements Y_1 and Y_2 that preserves the subspaces $\langle Y_3\rangle$ and $\langle Y_4\rangle$. This reduces the list of 15 gradings down to 11 distinct equivalence classes. Universal realizations for each equivalence class of torsion-free gradings are listed in Table 3.

rank	type	layers
3	(4)	$V_{1,0,0} \oplus V_{0,1,0} \oplus V_{1,1,0} \oplus V_{0,0,1} = \langle Y_1\rangle \oplus \langle Y_2\rangle \oplus \langle Y_3\rangle \oplus \langle Y_4\rangle$
2	(2, 1)	$V_{0,0} \oplus V_{1,0} \oplus V_{0,1} = \langle Y_2\rangle \oplus \langle Y_4\rangle \oplus \langle Y_1, Y_3\rangle$
2	(2, 1)	$V_{1,0} \oplus V_{0,1} \oplus V_{0,2} = \langle Y_4\rangle \oplus \langle Y_1, Y_2\rangle \oplus \langle Y_3\rangle$
2	(2, 1)	$V_{1,0} \oplus V_{0,1} \oplus V_{1,1} = \langle Y_1, Y_4\rangle \oplus \langle Y_2\rangle \oplus \langle Y_3\rangle$
2	(2, 1)	$V_{1,-1} \oplus V_{0,1} \oplus V_{1,0} = \langle Y_1\rangle \oplus \langle Y_2\rangle \oplus \langle Y_3, Y_4\rangle$
1	(0, 2)	$V_0 \oplus V_1 = \langle Y_1, Y_4\rangle \oplus \langle Y_2, Y_3\rangle$
1	(0, 2)	$V_1 \oplus V_2 = \langle Y_1, Y_2\rangle \oplus \langle Y_3, Y_4\rangle$
1	(1, 0, 1)	$V_1 \oplus V_2 = \langle Y_1, Y_2, Y_4\rangle \oplus \langle Y_3\rangle$
1	(1, 0, 1)	$V_0 \oplus V_1 = \langle Y_1, Y_2, Y_3\rangle \oplus \langle Y_4\rangle$
1	(1, 0, 1)	$V_0 \oplus V_1 = \langle Y_1\rangle \oplus \langle Y_2, Y_3, Y_4\rangle$
0	(0, 0, 0, 1)	$V_0 = \langle Y_1, Y_2, Y_3, Y_4\rangle$

TABLE 3. Gradings of the Lie algebra $L_{4,2}$

4.3. **Enumerating Heintze groups.** In this section, we present how our complete list of gradings for a given nilpotent Lie algebra $\mathfrak{g}$ can be used to determine a list of Heintze groups over $\mathfrak{g}$.

Definition 4.6. A *Heintze group* is a simply connected Lie group over $\mathbb{R}$ whose Lie algebra is a semidirect product of a nilpotent Lie algebra $\mathfrak{g}$ and $\mathbb{R}$ via a derivation $\alpha \in \mathrm{der}(\mathfrak{g})$ whose eigenvalues have strictly positive real parts.

Positive gradings for a given Lie algebra are naturally identified with diagonalizable derivations with strictly positive eigenvalues, see Subsection 2.3. Hence, to any positively graded Lie algebra $\mathfrak{g}$ we may associate a Heintze group over $\mathfrak{g}$. We shall call these groups *diagonal Heintze groups.*

The quasi-isometric classification of Heintze groups reduces to the study of so called *purely real Heintze groups*, for which the associated derivation has real eigenvalues. Note that diagonal Heintze groups form a subset of purely real Heintze groups. By [CPS17] the quasi-isometric classification problem of diagonal Heintze groups reduces to an algebraic problem of finding all the possible derivations defining non-isomorphic Heintze groups. Proposition 4.7 is a tool for tackling this algebraic problem using positive gradings. We will prove this result later in this section after discussing its role in the enumeration of Heintze groups.

Proposition 4.7. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\alpha, \beta \in \mathrm{der}(\mathfrak{g})$ diagonalizable derivations with strictly positive eigenvalues. If α and β define isomorphic Heintze groups, then they define equivalent $\mathbb{R}$-gradings.*

The enumeration of positive gradings we have established immediately gives the corresponding enumeration of diagonal Heintze groups over $\mathfrak{g}$. The enumeration of positive gradings can be understood in two different ways, see Remark 3.11. The corresponding enumeration of Heintze groups has similar character: it is either a parametrization via the projections or a finite list that does not contain all the isomorphism classes of Heintze groups but a representative for each family in terms of the layers. If one is able to eliminate equivalent gradings from the enumeration of positive gradings, then by Proposition 4.7 the corresponding list of Heintze groups does not contain isomorphic Heintze groups.

Remark 4.8. The enumeration of Heintze groups has a few caveats:

(i) Already over $\mathfrak{g} = \mathbb{R}^2$ there are uncountably many isomorphism classes of Heintze groups due to the multitude of possible projections $\mathbb{Z}^2 \to \mathbb{R}$. The same is true more generally whenever the maximal torus has dimension at least 2.

(ii) Our methods are in general able to find maximal gradings only for Lie algebras over algebraically closed fields. On the contrary, the base field of Heintze groups is $\mathbb{R}$.

Before proving Proposition 4.7, we need the following lemmas.

Lemma 4.9. *Let $\mathfrak{g}$ be a Lie algebra. Let $\delta \in \operatorname{der}(\mathfrak{g})$ be a diagonalizable derivation and let $X \in \mathfrak{g}$ be an eigenvector of δ. Then*

$$\operatorname{Ad}_{\exp(X)} \circ \delta \circ \operatorname{Ad}_{\exp(-X)} = \delta - \operatorname{ad}_{\delta(X)} .$$

Proof. Let $Y_1, \ldots, Y_n$ be a basis of $\mathfrak{g}$ that diagonalizes δ. Fix some $Y = Y_i$ and let w_Y and w_X be the eigenvalues of the eigenvectors Y and X. Since δ is a derivation and X and Y are eigenvectors, the vectors $\operatorname{ad}_X^k Y$ are also eigenvectors, and have the eigenvalues $kw_X + w_Y$. Using this fact, and recalling that $\operatorname{Ad}_{\exp(X)} = e^{\operatorname{ad}_X}$, see [Kna02, Proposition 1.91], we compute

$$\begin{aligned}
\delta \circ \operatorname{Ad}_{\exp(-X)}(Y) &= \delta\Big(\sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k Y\Big) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(kw_X + w_Y) \operatorname{ad}_{-X}^k Y \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \operatorname{ad}_{-X}^{k-1}[-w_X X, Y] + \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k (w_Y Y) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k(-[w_X X, Y] + w_Y Y) \\
&= \operatorname{Ad}_{\exp(-X)}(-\operatorname{ad}_{\delta(X)} Y + \delta(Y)).
\end{aligned}$$

By cancellation of $\operatorname{Ad}_{\exp(X)}$ and $\operatorname{Ad}_{\exp(-X)}$, the claimed formula follows. ∎

Lemma 4.10. *Let $X, Y \in \mathfrak{g}$ be two vectors of a Lie algebra $\mathfrak{g}$. Then*

$$\mathrm{Ad}_{\exp(X)} \circ \mathrm{ad}_Y \circ \mathrm{Ad}_{\exp(-X)} = \mathrm{ad}_{\mathrm{Ad}_{\exp(X)} Y} .$$

Proof. Since the map $\mathrm{Ad}_{\exp(X)}$ is a Lie algebra homomorphism and is the inverse of $\mathrm{Ad}_{\exp(-X)}$, we have

$$\mathrm{Ad}_{\exp(X)}[Y, \mathrm{Ad}_{\exp(-X)} Z] = [\mathrm{Ad}_{\exp(X)} Y, Z]$$

for every $Z \in \mathfrak{g}$. □

Lemma 4.11. *Let $\delta \in \mathrm{der}(\mathfrak{g})$ be a diagonalizable derivation with all eigenvalues strictly positive. Then for every vector $Y \in \mathfrak{g}$ there exists a vector $X \in \mathfrak{g}$ such that $\mathrm{Ad}_{\exp(X)} \circ \delta \circ \mathrm{Ad}_{\exp(-X)} = \delta - \mathrm{ad}_Y$.*

Proof. For a vector $X \in \mathfrak{g}$, denote by $C_X \colon \mathrm{der}(\mathfrak{g}) \to \mathrm{der}(\mathfrak{g})$ the conjugation map

$$C_X(\eta) = \mathrm{Ad}_{\exp(X)} \circ \eta \circ \mathrm{Ad}_{\exp(-X)} .$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ that diagonalizes δ. Consider the map

$$\Phi \colon \mathbb{R}^n \to \mathrm{der}(\mathfrak{g}), \quad \Phi(x_1, \ldots, x_n) = C_{x_n X_n} \circ \cdots \circ C_{x_1 X_1}(\delta).$$

By repeated application of Lemma 4.9 and Lemma 4.10, it follows that $\Phi(x) = \delta - \mathrm{ad}_{\phi(x)}$, where $\phi \colon \mathbb{R}^n \to \mathfrak{g}$ is the map

$$\begin{aligned}\phi(x_1, \ldots, x_n) = \delta(x_n X_n) &+ \mathrm{Ad}_{\exp(x_n X_n)} \delta(x_{n-1} X_{n-1}) + \cdots \\ &+ \mathrm{Ad}_{\exp(x_n X_n)} \mathrm{Ad}_{\exp(x_{n-1} X_{n-1})} \cdots \mathrm{Ad}_{\exp(x_2 X_2)} \delta(x_1 X_1).\end{aligned} \tag{6}$$

Since the composition of conjugations is a conjugation, it suffices to prove that the map ϕ is surjective.

Let $w_1, \ldots, w_n > 0$ be the eigenvalues of the vectors $X_1, \ldots, X_n$ for the derivation δ. Since the maps $x_i \mapsto \mathrm{sign}(x_i) |x_i|^{w_i}$ are all invertible, the map $\phi \colon \mathbb{R}^n \to \mathfrak{g}$ is surjective if and only if the map $\tilde{\phi} \colon \mathbb{R}^n \to \mathfrak{g}$ defined by

$$\tilde{\phi}(x_1, \ldots, x_n) = \phi(\mathrm{sign}(x_1) |x_1|^{w_1}, \ldots, \mathrm{sign}(x_n) |x_n|^{w_n}) \tag{7}$$

is surjective.

Let $D_\lambda \in \mathrm{Aut}(\mathfrak{g})$, $\lambda > 0$, be the one-parameter family of dilations defined by the derivation δ. Then for each $i = 1, \ldots, n$ the dilation is given by $D_\lambda(X_i) = \lambda^{w_i} X_i$ and we have the dilation equivariance

$$\mathrm{Ad}_{\exp(\lambda^{w_i} X_i)} \circ D_\lambda = D_\lambda \circ \mathrm{Ad}_{\exp(X_i)} .$$

Applying the above equivariance to the definition (7) we find that the map $\tilde{\phi}$ is D_λ-homogeneous, i.e., $\tilde{\phi}(\lambda x) = D_\lambda(\tilde{\phi}(x))$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Consequently the map $\tilde{\phi}$ is surjective if and only if it is open at zero. Since the change of parameters in (7) is a homeomorphism, the same is true also for the map ϕ.

By the definition (6), the map ϕ is smooth. The derivative of each summand $\mathrm{Ad}_{\exp(x_n X_n)} \cdots \mathrm{Ad}_{\exp(x_{i+1} X_{i+1})} \delta(x_i X_i)$ at zero is the map $x \mapsto \delta(x_i X_i)$, so the derivative $D_0 \phi$ of the map ϕ at zero is

$$D_0 \phi(x_1, \ldots, x_n) = \delta(x_1 X_1 + \cdots + x_n X_n).$$

By the strictly positive eigenvalue assumption, the map δ is invertible. Since $X_1, \ldots, X_n$ is a basis of $\mathfrak{g}$, it follows that the map ϕ is open at zero, concluding the proof. □

Proof of Proposition 4.7. Rescaling the derivations by a scalar, we may assume the smallest of the eigenvalues for both the derivations to be 1. Since the Heintze groups are assumed to be isomorphic, it is straightforward to see that there is a vector $X \in \mathfrak{g}$ so that the derivation α is conjugate by a Lie algebra automorphism of $\mathfrak{g}$ to the derivation $\beta + \mathrm{ad}_X$. We use Lemma 4.11 to find that actually α and β are conjugate. Applying Lemma 2.19(i) to the split tori spanned by α and β gives the desired result. □

The following example shows that Lemma 4.11 is false for some derivations with eigenvalues of different signs.

Example 4.12. Consider the Engel Lie algebra, i.e., the 4-dimensional Lie algebra given by the basis X_1, X_2, X_3, X_4 satisfying the bracket relations

$$[X_1, X_2] = X_3\ ,\ [X_1, X_3] = X_4.$$

Consider in this basis the derivations $\alpha = \mathrm{diag}(-3, 1, -2, -1)$ and $\beta = \alpha + \mathrm{ad}_{X_1}$. Then β is similar to α, but the similarity is not an automorphism.

4.4. **Bounds for non-vanishing $\ell^{q,p}$ cohomology.** Knowing all the possible positive gradings of a nilpotent Lie algebra $\mathfrak{g}$ has one further application in the realm of quasi-isometric classifications. Different positive gradings can be used to obtain better estimates in the computation of the $\ell^{q,p}$ cohomology of a nilpotent Lie group, which is a well-known quasi-isometry invariant.

By definition, the $\ell^{q,p}$ cohomology of a Riemannian manifold with bounded geometry is the $\ell^{q,p}$ cohomology of every bounded geometry simplicial complex quasi-isometric to it. A crucial result of [PR18] shows that in the case of contractible Lie groups, the $\ell^{q,p}$ cohomology of the manifold is isomorphic to its $L^{q,p}$ cohomology.

Definition 4.13. The $L^{q,p}$ *cohomology* of a nilpotent Lie group G is defined as

$$L^{q,p}H^\bullet(G) = \frac{\{\text{closed forms in } L^p\}}{d\big(\{\text{forms in } L^q\}\big) \cap L^p}.$$

In [PR18, Theorem 1.1] it is shown that the Rumin complex constructed on a Carnot group allows for sharper computations regarding $L^{q,p}H^\bullet(G)$ when compared to the usual de Rham complex. Defining and reviewing the properties of the Rumin complex $(E_0^\bullet, d_c)$ goes beyond the scope of this paper. For the following discussion, it is sufficient to know that the space of Rumin h-forms E_0^h is a subspace of the space of smooth differential h-forms of the underlying nilpotent Lie group G.

Definition 4.14. Let us consider a positive grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in \mathbb{R}} V_\alpha$. Then a left-invariant 1-form θ has *weight* α, that is $w(\theta) = \alpha$, if $\theta = X^*$ for $X \in V_\alpha$. In other words, θ is the dual of a vector field belonging to the subspace V_α at the identity of the group. In general, given a left-invariant h-form, we will say that it has *weight* p if it can be expressed as a linear combination of left-invariant h-forms $\theta_{i_1,\dots,i_h} = \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ such that $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = p$.

Given a positive grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in \mathbb{R}} V_\alpha$, we call the quantity

$$Q = \sum_{\alpha \in \mathbb{R}_+} \alpha \dim V_\alpha$$

the *homogeneous dimension* of $\mathcal{V}$. We also define for each degree h the number

$$\delta N_{\min}(h) = \min_{\theta \in E_0^h} w(\theta) - \max_{\tilde{\theta} \in E_0^{h-1}} w(\tilde{\theta}) .$$

The following is [PR18, Theorem 1.1(ii)].

Theorem 4.15. *Let G be a Carnot group and homogeneous dimension Q. If*

$$1 \leq p, q \leq \infty \text{ and } \frac{1}{p} - \frac{1}{q} < \frac{\delta N_{\min}(h)}{Q}$$

then the $L^{q,p}$ cohomology of G in degree h does not vanish.

Moreover, in Theorem 9.2 of the same paper it is shown how the non-vanishing statement has a wider scope, as it can be applied to Carnot groups equipped with a homogeneous structure that comes from a positive grading. This result has been further extended in [Tri20] to arbitrary positively graded nilpotent Lie groups.

A natural question that stems from these considerations is whether it is possible to identify which choice of positive grading will yield the best interval for non-vanishing cohomology. This problem can be easily presented in terms of maximising the value of the fraction $\delta N_{\min}(h)/Q$ among all the possible positive gradings for a given Lie group G.

Let us describe the maximization procedure in more detail. Let $\mathcal{W} : \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}^k} W_n$ be a maximal grading of $\mathfrak{g}$ and let Ω be the set of weights of $\mathcal{W}$. For $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$, let $\pi^{\mathbf{a}} \colon \mathbb{Z}^k \to \mathbb{R}$ be the projection given by $\pi^{\mathbf{a}}(e_i) = a_i$. Let

$$\mathbf{A}_+ = \{\mathbf{a} \in \mathbb{R}^k : \pi^{\mathbf{a}}(n) > 0 \, \forall n \in \Omega\}. \tag{8}$$

The push-forward $\pi^{\mathbf{a}}_*(\mathcal{W})$ is a positive grading if and only if $\mathbf{a} \in \mathbf{A}_+$.

In the sequel we shall identify any positive grading of $\mathfrak{g}$ with the corresponding vector $\mathbf{a} \in \mathbf{A}_+$. In particular, if θ is the dual of $X \in W_n$, then the weight of θ with respect to the grading $\pi^{\mathbf{a}}_*(\mathcal{W})$ is $w(\theta)_{\mathbf{a}} =$

$\pi^{\mathbf{a}}(n)$. Then we want to find the value of the following expression for each degree h:

$$\max_{\mathbf{a}\in\mathbf{A}_+}\left\{\frac{\min_{\theta\in E_0^h} w(\theta)_{\mathbf{a}} - \max_{\tilde{\theta}\in E_0^{h-1}} w(\tilde{\theta})_{\mathbf{a}}}{Q_{\mathbf{a}}}\right\},$$

where $Q_{\mathbf{a}}$ is the homogeneous dimension of $\pi_*^{\mathbf{a}}(\mathcal{W})$.

A problem of this form can be converted into a linear optimization problem as follows:

1. replace $\min_{\theta\in E_0^h} w(\theta)_{\mathbf{a}}$ with a new variable x, and add the constraint $x \le w(\theta)_{\mathbf{a}}$ for each $\theta \in E_0^h$;
2. replace $\max_{\tilde{\theta}\in E_0^{h-1}} w(\tilde{\theta})$ with a new variable y, and add the constraint $y \ge w(\tilde{\theta})_{\mathbf{a}}$ for each $\tilde{\theta} \in E_0^{h-1}$;
3. normalize the expression by imposing $Q_{\mathbf{a}} = 1$.

We are then left with the following expression for our original maximization problem

$$\begin{aligned} \text{Maximize} \quad & x - y \\ \text{subject to} \quad & x \le w(\theta)_{\mathbf{a}} \quad \forall\, \theta \in E_0^h, \\ & y \ge w(\tilde{\theta})_{\mathbf{a}} \quad \forall \tilde{\theta} \in E_0^{h-1}, \\ & Q_{\mathbf{a}} = 1, \quad \mathbf{a} \in \mathbf{A}_+ \end{aligned}$$

which can easily be solved by a computer, yielding the optimal bound for non-vanishing cohomology using the method of Theorem 4.15.

Example 4.16. Let us consider the non-stratifiable Lie group G of dimension 6, whose Lie algebra is denoted as $L_{6,10}$ in [dG07], with the non-trivial brackets

$$[X_1, X_2] = X_3\ ,\ [X_1, X_3] = [X_5, X_6] = X_4.$$

The space of Rumin forms in G is

$$\begin{aligned} E_0^1 &= \langle \theta_1, \theta_2, \theta_5, \theta_6 \rangle; \\ E_0^2 &= \langle \theta_{5,6} - \theta_{1,3}, \theta_{1,5}, \theta_{1,6}, \theta_{2,3}, \theta_{2,5}, \theta_{2,6} \rangle; \\ E_0^3 &= \langle \theta_{2,5,6} + \theta_{1,2,3}, \theta_{2,3,5}, \theta_{2,3,6}, \theta_{1,3,4} - \theta_{4,5,6}, \theta_{1,4,5}, \theta_{1,4,6} \rangle. \end{aligned}$$

For the Lie algebra $L_{6,10}$, the maximal grading is over $\mathbb{Z}^3$ with the layers

$$\begin{aligned} V_{(0,1,0)} &= \langle X_1 \rangle, & V_{(0,0,1)} &= \langle X_2 \rangle, & V_{(0,1,1)} &= \langle X_3 \rangle \\ V_{(0,2,1)} &= \langle X_4 \rangle, & V_{(1,0,0)} &= \langle X_5 \rangle, & V_{(-1,2,1)} &= \langle X_6 \rangle. \end{aligned}$$

The family of projections $\pi^{\mathbf{a}}\colon \mathbb{Z}^3 \to \mathbb{R}$ giving positive gradings is parametrized by $(a_1, a_2, a_3) = \mathbf{a} \in \mathbf{A}_+$ as in (8). The weights of left-invariant 1-forms are

$$\begin{aligned} w(\theta_1)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,1,0) = a_2;\\ w(\theta_2)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,0,1) = a_3;\\ w(\theta_3)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,1,1) = a_2 + a_3;\\ w(\theta_4)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,2,1) = 2a_2 + a_3;\\ w(\theta_5)_{\mathbf{a}} &= \pi^{\mathbf{a}}(1,0,0) = a_1;\\ w(\theta_6)_{\mathbf{a}} &= \pi^{\mathbf{a}}(-1,2,1) = 2a_2 + a_3 - a_1. \end{aligned}$$

From this computation we get the explicit expression

$$\mathbf{A}_+ = \{\mathbf{a} \in \mathbb{R}^3 : a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ -a_1 + 2a_2 + a_3 > 0\}$$

and the homogeneous dimension $Q_{\mathbf{a}} = 6a_2 + 4a_3$.

Let us first consider the bound for non-vanishing cohomology in degree 1. We express

$$\max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\delta N_{\min}(1)}{Q_{\mathbf{a}}} \right\} = \max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\min\{a_1, a_2, a_3, 2a_2 + a_3 - a_1\}}{6a_2 + 4a_3} \right\}.$$

as the linear optimization problem

$$\begin{aligned} \text{Maximize}\quad & x\\ \text{subject to}\quad & x \le a_1,\ x \le a_2,\ x \le a_3,\\ & x \le 2a_2 + a_3 - a_1,\\ & 1 = 6a_2 + 4a_3,\\ & a_1, a_2, a_3 > 0,\ 2a_2 + a_3 - a_1 > 0. \end{aligned}$$

A solver finds the solution $\frac{1}{10}$, which is obtained by choosing $a_1 = a_2 = a_3 = \frac{1}{10}$. Since the quantity $\frac{\delta N_{\min}(1)}{Q_{\mathbf{a}}}$ is scaling invariant, we find that the grading defined by $a_1 = a_2 = a_3 = 1$ gives $\ell^{q,p}H^1(G) \neq 0$ with the optimal bound $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Similarly, once we re-express

$$\max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\delta N_{\min}(2)}{Q_{\mathbf{a}}} \right\}.$$

as a linear optimization problem and feed it into a solver, we get the result $\frac{1}{10}$, obtained (up to rescaling) by taking $a_2 = a_3 = 2$ and $a_1 = 3$. Therefore $\ell^{q,p}H^2(G) \neq 0$ for $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Likewise, we obtain the optimal bound $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$ for $\ell^{q,p}H^3(G) \neq 0$ by taking $a_1 = a_2 = a_3 = 1$.

Finally, by Hodge duality, see [Tri20, Theorem 7.3], we obtain the optimal bounds for $\ell^{q,p}$ cohomology in complementary degree, that is $\ell^{q,p}H^4(G) \neq 0$, $\ell^{q,p}H^5(G) \neq 0$, and $\ell^{q,p}H^6(G) \neq 0$, for $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Remark 4.17. [PR18, Example 9.5] describes an explicit positive grading in the Engel group that gives an improved bound for the non-vanishing of the $L^{q,p}$ cohomology in degree 2. By a similar computation as the one shown in Example 4.16, one can verify that the value given in [PR18, Example 9.5] is indeed the optimal bound.

Appendix A. Existence of a positive realization

Example 2.4 motivates an alternate approach for deciding the existence of a positive realization. The grading in the example does not admit a positive realization: suppose by contradiction that there is an injection $\{a, b, c, d\} \to \mathbb{R}$ that gives a positive realization. The bracket relations of the Lie algebra imply the equations $a + c = d$ and $b + d = c$, which are impossible for strictly positive weights. Here the non-existence of a positive realization is found simply by considering the equations implied by the bracket relations of the layers.

In general, a grading can be realized over some abelian group A with the set of weights $\{\lambda_1, \dots, \lambda_k\}$ if and only if certain system of equations of the type $\lambda_i + \lambda_j = \lambda_h$ has a solution $(\lambda_1, \dots, \lambda_k)$ whose components are all distinct. This system consists of equations coming from the non-trivial bracket relations among the layers of the grading, see step 2 of Algorithm 2.13. A positive realization exists if and only if there is a solution in the group $A = \mathbb{R}$ with all weights strictly positive. Indeed, if there is a positive solution, then there is also a positive solution with distinct components as we shall see in the proof of Algorithm A.4.

Deciding if a solution exists with all components strictly positive is a classical problem in linear programming. By rescaling, we may replace the open conditions $\lambda_i > 0$ with the closed conditions $\lambda_i \geq 1$. By a change of variables $\mu_i = \lambda_i - 1$, we find that the linear problem for the existence of a positive realization is equivalent to an affine problem $\mu_i + \mu_j - \mu_h = -1$ with all the components μ_i non-negative.

Let A be the $N \times k$-matrix of coefficients of the affine problem and let $\mathbf{b} = (-1, \dots, -1) \in \mathbb{R}^N$. By getting rid of linearly dependent equations, we may assume that $\operatorname{rank}(A) = N \leq k$. Our goal is then an algorithm that either produces an element of the set

$$P = \{\mathbf{x} \in \mathbb{R}^k \mid A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0\} \tag{9}$$

or indicates that the set P is empty. Here we use the shorthand notation $\mathbf{x} \geq 0$ to mean that all the components of the vector $\mathbf{x}$ are non-negative. Notice that the set P of solutions is closed and convex.

There is a vast literature on how to solve linear programming problems, and we refer to the book [FP93]. This approach and in particular Lemma A.3 below are essentially from section 2.4 of that book. We state them here for completeness.

Definition A.1. Let $K \subset \mathbb{R}^n$ be a convex set. We say that a point $x \in K$ is an *extremal point* if it cannot be expressed as a non-trivial convex combination of the points of K.

Lemma A.2. *Let $K \subset \mathbb{R}^n$ be a closed convex non-empty set for which $\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$. Then K contains at least one extremal point.*

Proof. Consider the lexicographic order $\prec$ on $\mathbb{R}^n$, where $\mathbf{x} \prec \mathbf{y}$ if there exists some index $i \in \{1, \dots, n\}$ such that $\mathbf{x}_j = \mathbf{y}_j$ for all $j < i$ and $\mathbf{x}_i < \mathbf{y}_i$. Observe that if $\mathbf{x} \prec \mathbf{y}$, then

$$\mathbf{x} \prec t\mathbf{x} + (1-t)\mathbf{y} \tag{10}$$

for every $0 < t < 1$. Since $\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$, there exists a lexicographic minimum $\mathbf{x}_{\min} \in K$. It follows from (10) that the point $\mathbf{x}_{\min}$ cannot be expressed as a non-trivial convex combination. □

Lemma A.3. *Let P be the set of non-negative solutions of a system $A\mathbf{x} = \mathbf{b}$ as in (9). Let $\mathbf{x} \in \mathbb{R}^k$ be an extremal point of P. Then there exists an invertible $N \times N$ matrix B whose columns are chosen from the matrix A, such that up to a permutation of components, $\mathbf{x} = (B^{-1}(\mathbf{b}), 0, \dots, 0) \in \mathbb{R}^k$.*

Proof. By permuting the basis, we can express $\mathbf{x} = (\mathbf{x}_+, 0, \dots, 0)$, where $\mathbf{x}_+ \in \mathbb{R}^p$ for some $p \leq k$ and $\mathbf{x}_+ > 0$. Let A_+ be the matrix consisting of the first p columns of A. First we show that the matrix A_+ has rank p. Suppose towards a contradiction that there is a non-zero vector $\mathbf{w} \in \mathbb{R}^p$ such that $A_+\mathbf{w} = 0$. Let $\delta > 0$ be so small that the vectors

$$\mathbf{z}_1 = \mathbf{x}_+ + \delta\mathbf{w} \qquad \mathbf{z}_2 = \mathbf{x}_+ - \delta\mathbf{w}$$

both satisfy $\mathbf{z}_1, \mathbf{z}_2 \geq 0$. For both $i \in \{1, 2\}$, let $\mathbf{u}_i = (\mathbf{z}_i, 0, \dots, 0) \in \mathbb{R}^k$. Then

$$A\mathbf{u}_i = A_+\mathbf{z}_i = A_+\mathbf{x}_+ = A\mathbf{x} = \mathbf{b},$$

so $\mathbf{u}_i \in P$ are both solutions. Now the solution $\mathbf{x}$ can be represented as a non-trivial convex combination $\mathbf{x} = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$ of solutions, which contradicts the assumption that $\mathbf{x}$ is an extremal point. We conclude that the rank of A_+ must be p. Since $\mathrm{rank}(A) = N$, we deduce also $p \leq N$.

If $p < N$, then since $\mathrm{rank}(A) = N$ it is possible to form an invertible matrix B by adding some further columns of A to the matrix A_+. If instead $p = N$, then continue with $B = A_+$. Let $\bar{\mathbf{x}} = (\mathbf{x}_+, 0, \dots, 0) \in \mathbb{R}^N$. Then

$$B\bar{\mathbf{x}} = A_+\mathbf{x}_+ = A\mathbf{x} = \mathbf{b}$$

so $\bar{\mathbf{x}} = B^{-1}(\mathbf{b})$ and the claim follows. □

Algorithm A.4 (Existence of a positive realization)**.** *Input: A grading $\mathcal{V}$ of a Lie algebra $\mathfrak{g}$. Output: Decision if $\mathcal{V}$ admits a positive realization.*

(1) Form the $N \times k$ matrix A associated with the problem and set $\mathbf{b} = (-1, \dots, -1) \in \mathbb{R}^N$.
(2) For each invertible $N \times N$ matrix B formed from the columns of the matrix A do the following: Compute $\mathbf{x} = B^{-1}(\mathbf{b})$. If $\mathbf{x} \geq 0$, then the grading admits a positive realization.
(3) Otherwise, the grading has no positive realization.

Proof of correctness. Let P be as in (9). Lemma A.3 implies that step 2 constructs all the extremal points of P. By Lemma A.2, if no extremal points are found, the set P is empty and no positive realization exists. We still need to argue that if P is non-empty, then a positive realization exists.

A priori, even if some $\mathbf{x} = B^{-1}(\mathbf{b}) \geq 0$, the corresponding weights $x_i + 1, \dots, x_k + 1$ do not necessarily define a realization of the original grading, since in general these weights are non-distinct and hence define a coarser grading. By Lemma 2.15 this coarser grading is a push-forward grading of the universal realization of the original grading, via a homomorphism from some $\mathbb{Z}^m$ to $\mathbb{R}$ mapping the weights to $x_1 + 1, \dots, x_k + 1$. The homomorphism is realized as a projection to some line of $\mathbb{R}^m$, as in the proof of Lemma 3.7. By perturbing this line, it is always possible to find another homomorphism that is injective on the weights and maps all the weights to strictly positive reals. Hence there is also a positive realization of the original grading. □

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