Tangent and asymptotic cones of geodesics

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Introduction

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Are geodesics smooth? Are they even differentiable?

A priori geodesics are Lipschitz, so at least they are differentiable almost everywhere. Beyond this, little is known in the general case.

Introduction

We approach the differentiability problem from a metric geometry viewpoint through the infinitesimal geometry of a sub-Riemannian manifold.

Theorem (Mitchell 1985)

The metric tangent of an equiregular sub-Riemannian manifold is a sub-Riemannian Carnot group.

Theorem (Bellaïche 1996)

The metric tangent of any sub-Riemannian manifold is a sub-Riemannian homogeneous space, that is, a quotient of a sub-Riemannian Carnot group.

Introduction

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Due to the self-similarity of G by dilations, the infinitesimal geometry is given by the Carnot group G itself, which lends itself to a nice metric characterization of differentiability:

Lemma

A curve $\gamma : I \to G$ is differentiable at $t \in I$ if and only the tangent cone of γ at t consists of a single line.

Tangent cones

Tangent cones

Let $\gamma : I \to G$ be a curve. To define the tangent cone of γ at t_0, we study dilated copies of the curve centered at $\gamma(t_0)$.

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For any dilation factor $h > 0$, define $I_h = \frac{1}{h}(I - t_0)$ and

$$\gamma_h : I_h \to G, \quad \gamma_h(t) = \delta_{\frac{1}{h}} \left(\gamma(t_0)^{-1} \gamma(t_0 + ht) \right).$$

This is simply the non-abelian version of the difference quotient used in the definition of derivatives.

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If the limit $\lim_{h\to 0} \gamma_h(1)$ exists, it is the Pansu-derivative of γ at t_0, and in particular the curve is differentiable at t_0.

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In general, there is no need for $\lim_{h \to 0} \gamma_h$ to exist.

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If γ is L-Lipschitz, then so is γ_h. Thus $\{\gamma_h : \ h > 0\}$ is a family of L-Lipschitz curves in G with $\gamma_h(0) = e$ for all $h > 0$.

Ascoli-Arzelá $\implies$ for every sequence $h_j \to 0$ there is a subsequence h_{j_k} and a curve $\sigma : \mathbb{R} \to G$ such that $\gamma_{h_{j_k}} \to \sigma$ uniformly on compact sets of $\mathbb{R}$.

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The tangent cone of γ at t_0 is the collection of all such curves:

$$\mathrm{Tang}(\gamma, t_0) = \left\{\sigma \mid \exists h_j \to 0 : \gamma_{h_j} \to \sigma\right\}.$$

Tangent cones

A simple result of metric geometry is:

Lemma

Let $\gamma : I \to G$ be a geodesic and $t \in I$. Then every $\sigma \in \mathrm{Tang}(\gamma, t)$ is also a geodesic in G.

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However, using the properties of Carnot groups we are able to prove something less trivial:

Theorem (H. – Le Donne)

Let $\gamma : I \to G$ be a geodesic and $t \in I$. Then for every $\sigma \in \mathrm{Tang}(\gamma, t)$, the curve $\pi_s \circ \sigma$ is a geodesic in the Carnot group $G/\exp(V_s)$ of one step lower.

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Here $V_1 \oplus \cdots \oplus V_s = \mathfrak{g}$ is the stratification of the Lie algebra of G and $\pi_s : G \to G/\exp(V_s)$ is the quotient projection.

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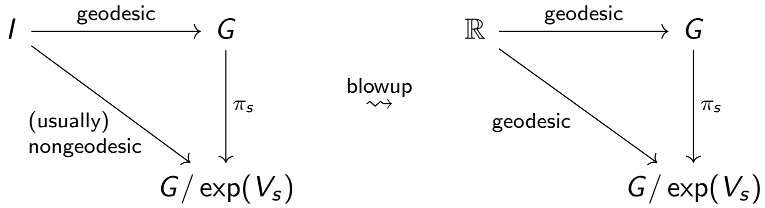

Tangent cones

Another useful simple result of metric geometry is that tangents of tangents are tangents:

Lemma

$\mathrm{Tang}(\mathrm{Tang}(\gamma, t), 0) \subset \mathrm{Tang}(\gamma, t)$.

The proof of this lemma is a diagonal argument using the continuity of dilations and the homomorphism property

$$\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}.$$

Tangent cones

Iterate the previous theorem:

γ

geodesic in

G

Tangent cones

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$$\gamma \quad \overset{\text{blowup}}{\rightsquigarrow} \quad \begin{array}{c} \sigma_1 \\ \pitchfork \\ \mathrm{Tang}(\gamma, t) \end{array}$$

geodesic in G	geodesic in $G/\exp(V_s)$

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$$\begin{array}{ccccc} \gamma & \overset{\text{blowup}}{\rightsquigarrow} & \sigma_1 & \overset{\text{blowup}}{\rightsquigarrow} & \sigma_2 \\ & & \Cap & & \Cap \\ & & \mathrm{Tang}(\gamma, t) & \supset & \mathrm{Tang}(\sigma_1, 0) \end{array}$$

geodesic in G	geodesic in $G/\exp(V_s)$	geodesic in $G/\exp(V_{s-1} \oplus V_s)$

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\gamma & \overset{\text{blowup}}{\rightsquigarrow} & \sigma_1 & \overset{\text{blowup}}{\rightsquigarrow} & \sigma_2 & \overset{\text{blowup}}{\rightsquigarrow} & \dots & \overset{\text{blowup}}{\rightsquigarrow} & \sigma_{s-1} \\
 & & \Cap & & \Cap & & & & \Cap \\
 & & \mathrm{Tang}(\gamma, t) & \supset & \mathrm{Tang}(\sigma_1, 0) & \supset & \dots & \supset & \mathrm{Tang}(\sigma_{s-2}, 0) \\
\text{geodesic in } G & & \text{geodesic in } G/\exp(V_s) & & \text{geodesic in } G/\exp(V_{s-1} \oplus V_s) & & \dots & & \text{geodesic in } G/[G, G]
\end{array}$$

$\implies$ When G is a step s Carnot group, any $s-1$ times iterated tangent of a geodesic is also geodesic in the horizontal space $G/[G, G]$.

Tangent cones

When G is sub-Riemannian, $G/[G, G]$ is an inner product space. The only geodesics in an inner product space are lines, so we get another proof of

Theorem (Monti–Pigati–Vittone 2017)

If $\gamma : I \to M$ is a geodesic in a sub-Riemannian manifold, then for every $t \in I$ there exists a line in $\mathrm{Tang}(\gamma, t)$.

Asymptotic cones

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$\Longrightarrow$ Knowledge about infinite geodesics is relevant also within the regularity problem.

Hence we apply our techniques also to the study of the large scale behavior of geodesics through their asymptotic cones:

$$\mathrm{Asymp}(\gamma) = \left\{\sigma \mid \exists h_j \to \infty : \gamma_{h_j} \to \sigma\right\}$$

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$$\mathrm{Asymp}(\gamma) = \left\{\sigma \mid \exists h_j \to \infty : \gamma_{h_j} \to \sigma\right\}$$
$$\mathrm{Tang}(\gamma, t_0) = \left\{\sigma \mid \exists h_j \to 0 : \gamma_{h_j} \to \sigma\right\}$$

Asymptotic cones

Theorem (H. – Le Donne)

If $\gamma : \mathbb{R} \to G$ is a geodesic, then

(i) $\pi \circ \gamma : \mathbb{R} \to G/[G,G]$ *is a geodesic,*

or

(ii) $\exists$ *a hyperplane* $W \subset G/[G,G]$ *and* $\exists R > 0$ *such that* $\pi \circ \gamma(\mathbb{R}) \subset B_{G/[G,G]}(W, R)$.

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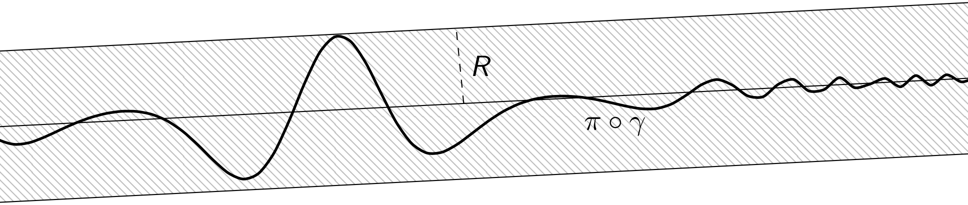

Asymptotic cones

Corollary

If G is sub-Riemannian, then for every geodesic $\gamma : \mathbb{R} \to G$ there exists a Carnot subgroup $H < G$ of lower rank such that

$$\sigma \in \mathrm{Asymp}(\gamma) \implies \sigma(\mathbb{R}) \subset H.$$

The subgroup H is the Carnot group generated by the horizontal hyperplane W.

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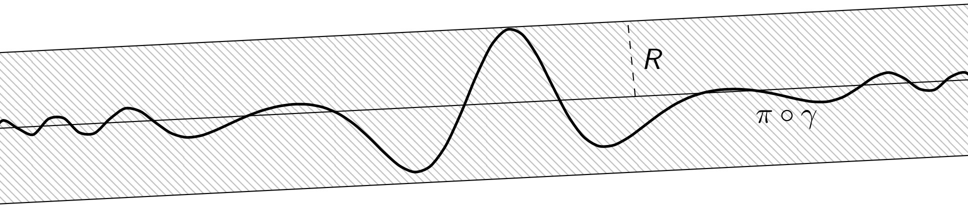

Asymptotic cones

Theorem (H. – Le Donne)

*If $\gamma : \mathbb{R} \to G$ is a $(1, C)$-**quasi**-geodesic, then*

(i) $\pi \circ \gamma : \mathbb{R} \to G/[G, G]$ *is a* $(1, \tilde{C})$-**quasi**-*geodesic,*

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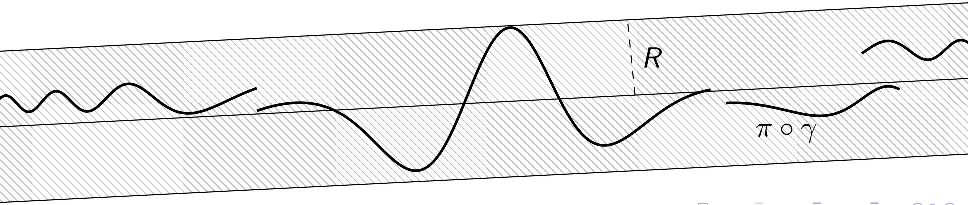

The core ideas of the proofs

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It is in principle simple to show that a curve is not a geodesic: find a shorter curve with the same endpoints.

Hence to prove properties of geodesics:

(1) Assume some property does *not* hold for an arbitrary curve.

(2) Use the assumption to construct a *shorter* curve with the same endpoints.

The core ideas of the proofs

The *cut & correct* strategy to shortening a curve:

(2a) The cut: replace some curve segment $\gamma|_{[a,b]}$ with the lift of a geodesic from $G/\exp(V_s)$, shortening γ by some $\epsilon > 0$, but changing its endpoint.

(2b) The correction: perturb the curve so that
  (i) the endpoint is reverted to the original endpoint, and
  (ii) length is increased by no more than ϵ.

The cut

From the algebraic viewpoint, lifting a geodesic can be rewritten as a two point lifting property:

Proposition

For any $g \in G$ *there exists* $h \in \exp(V_s)$ *such that*

$$d_{G/\exp(V_s)}(e, \pi_s g) = d_G(e, hg).$$

After replacing $\gamma|_{[a,b]}$ with a geodesic segment from $G/\exp(V_s)$, either

(i) we decrease length by $\epsilon > 0$, but the endpoint is left-translated by some $h \in \exp(V_s)$, or

(ii) $\pi_s \circ \gamma|_{[a,b]}$ was itself a geodesic, and the endpoint does not change.

The correction

Perturb the curve via the insertion of correcting curves.

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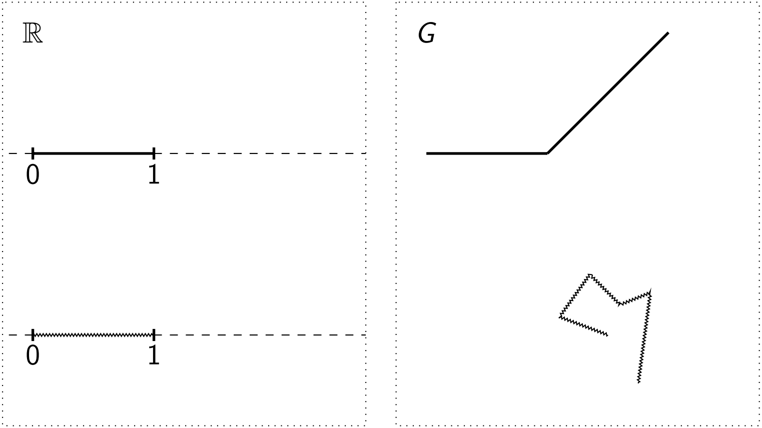

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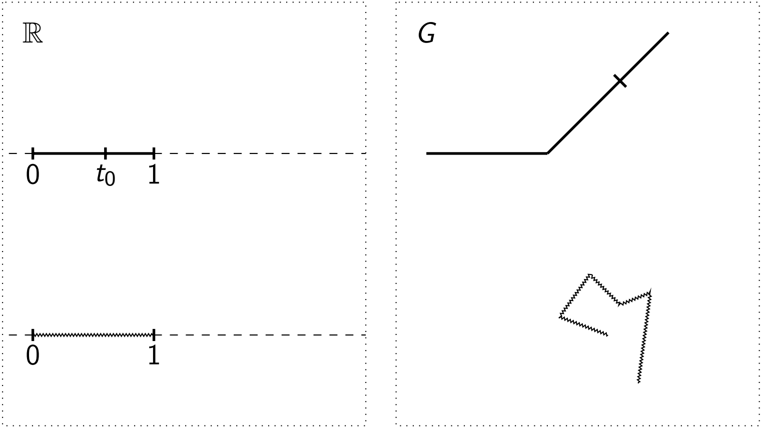

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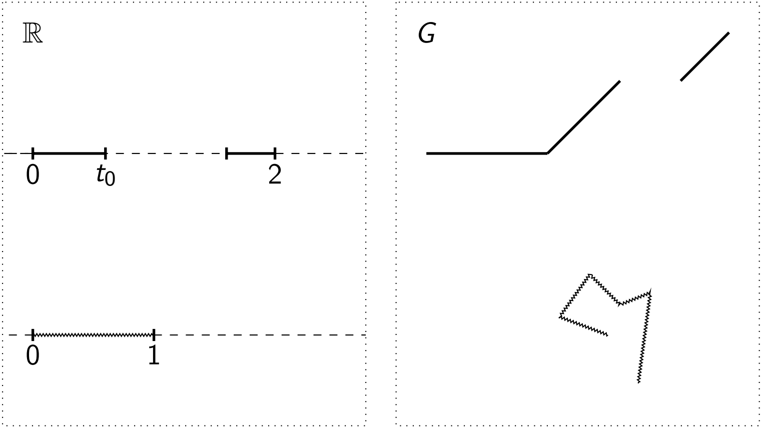

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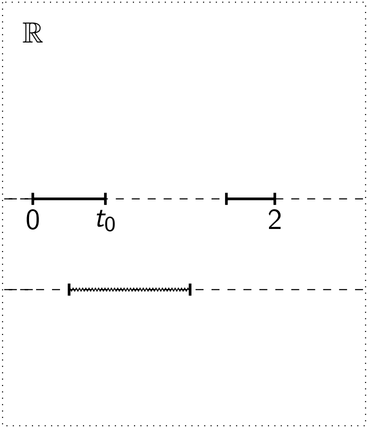

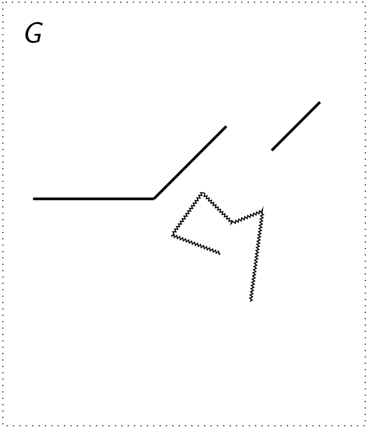

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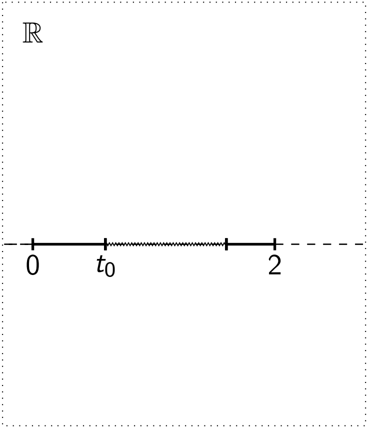

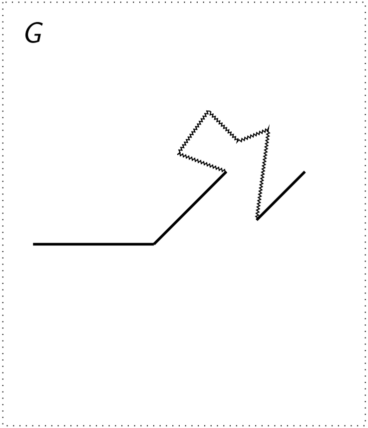

The correction

Algebraically, inserting a curve $\alpha : [0, 1] \to G$ at a point $g = \gamma(t)$ will left-translate the endpoint of the curve by

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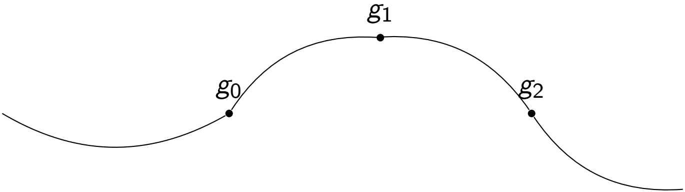
Idea: the insertion can change the endpoint by much more than the addition of length when $g = \gamma(t)$ is far from the identity.

The correction

More explicitly, the correction we use is as follows. Denote by r the rank of G.

(1) Choose $r+1$ points $g_0, \ldots, g_r$ along the curve γ.

(2) For each curve segment g_{k-1} to g_k, insert α_k at g_{k-1}, and insert the reverse α_k^{-1} at g_k.



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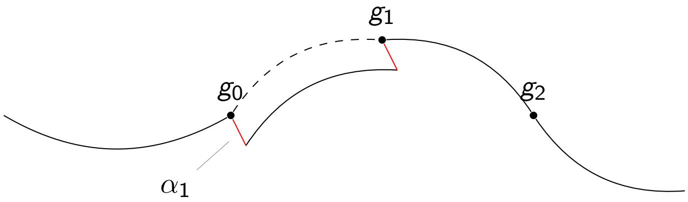

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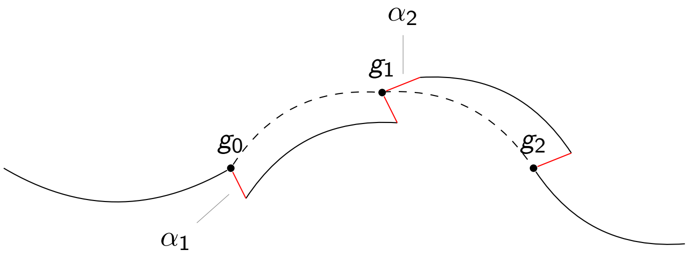

The correction

A back-and-forth perturbation is a group commutator:

$$a\alpha a^{-1} \cdot b\alpha^{-1} b = a[\alpha, a^{-1} b] a^{-1}.$$

$\implies$ Perturbation in the layer $s - 1$ correct an error in layer s.

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Linear algebra $\implies$ our error correction method is reduced to solving

$$L(\alpha_1, \dots, \alpha_r) = \log h,$$

where $L : (V_{s-1})^r \to V_s$ is a linear map depending on the points $g_0, \dots, g_r$.

Estimating the correction

Corrections in layer $s-1$
$\implies$ L only depends on the horizontal projections $\pi(g_0), \dots, \pi(g_r)$
$\implies$ $L(\alpha_1, \dots, \alpha_r) = \log h$ has a simple geometric description:

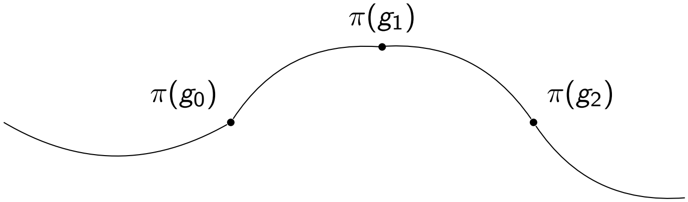
$$\|\alpha_k\| \lesssim \frac{\|\log h\|}{F(g_0, \dots, g_r)}.$$

where $F(g_0, \dots, g_r)$ is the smallest height of the parallelotope with sides

$$x_k = \pi(g_k) - \pi(g_{k-1}) \in G/[G,G], \quad k = 1, \dots, r$$

in the normed space $G/[G,G] \simeq (\mathbb{R}^r, \|\cdot\|)$.

Estimating the correction



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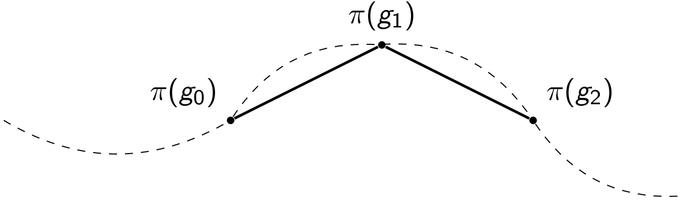

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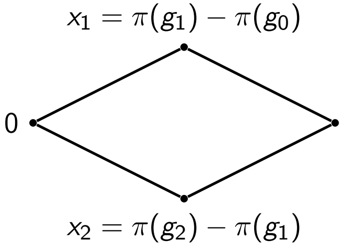

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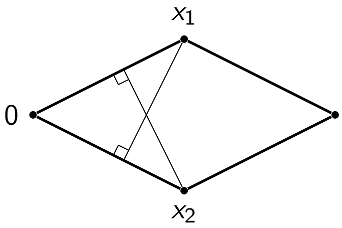

$$F(g_0, g_1, g_2) = \min\{d(x_1, \operatorname{span} x_2), d(x_2, \operatorname{span} x_1)\}.$$

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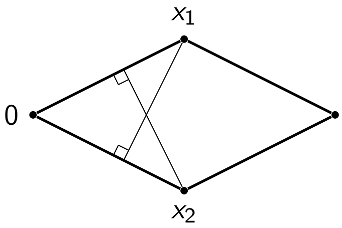

$$F(g_0, g_1, g_2) = \min\{d(x_1, \operatorname{span} x_2), d(x_2, \operatorname{span} x_1)\}.$$

$\implies$ the size of parallelotopes in the horizontal projection determines how large errors can be corrected.

Tangent cones

If γ has any non-degenerate parallelotope of size $F(g_0, \ldots, g_r) \geq R$, then γ_h has a parallelotope of size $\geq \dfrac{R}{h}$.

$R/h \to \infty$ as $h \to 0 \implies$ any cut of a tangent in $G/\exp(V_s)$ cannot gain *any* length.

Asymptotic cones

If γ is not a geodesic in $G/\exp(V_s)$, it must contain only parallelotopes of bounded size $F(g_0, \ldots, g_r) \leq M$.

By a Euclidean compactness argument, any set in $\mathbb{R}^r$ which contains only parallelotopes of size $\leq R$, is contained in a R-neighborhood of a hyperplane.

Thanks for your attention!