Non-minimality of corners in subriemannian geometry (joint work with E. Le Donne)

Eero Hakavuori

University of Jyväskylä

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Regularity of length-minimizers

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Toward which extreme does the subriemannian case tend to?

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Theorem (H. and Le Donne 2016)

Length-minimizing curves on subriemannian manifolds do not have corner-type singularities.

$\Longrightarrow$ A potential non-smooth geodesic must be more complicated than just a curve that has one-sided derivatives everywhere and is C^1 outside of a single point.

1 Regularity of length-minimizers

2 Reduction of the regularity problem to Carnot groups
- Desingularization
- Metric blow-up
- Rank reduction

3 Cutting corners in Carnot groups
- The Euclidean and Heisenberg cases
- Lifting curves from step $s - 1$ to step s
- Error correction

Regularity of length-minimizers

Almost all of the known regularity results for subriemannian geodesics are for specific types of subriemannian manifolds. For example

- Golé and Karidi 1995: Geodesics in step 2 Carnot groups are smooth.
- Leonardi and Monti 2008: Corners are not length-minimizing on equiregular subriemannian manifolds satisfying a condition on the iterated Lie-brackets of length ≥ 4.

Regularity of length-minimizers

One completely general result exists:

Theorem (Sussmann 2014)

On analytic subriemannian manifolds, any arc length parametrized length-minimizer is analytic on an open dense set.

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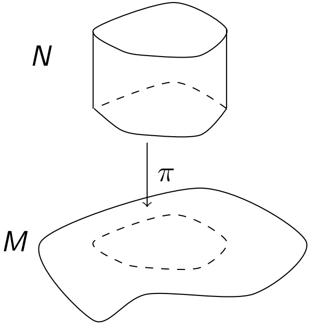

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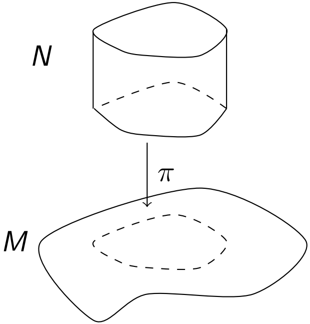
Lemma

Lifting curves preserves corner-type singularities and lengths of curves.



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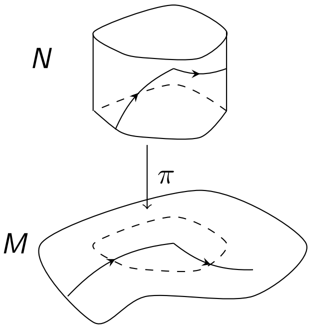

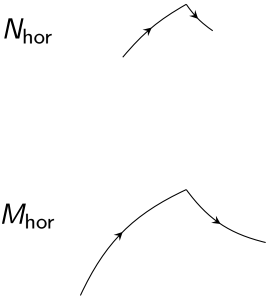

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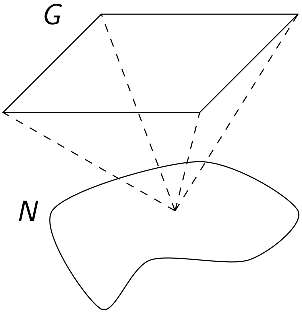

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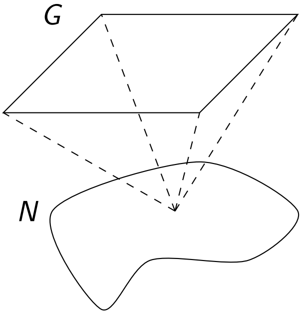

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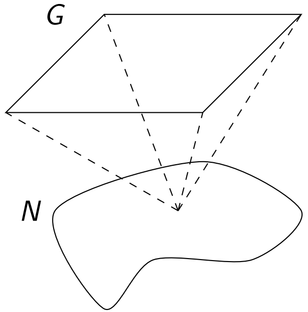

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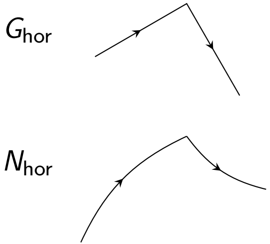
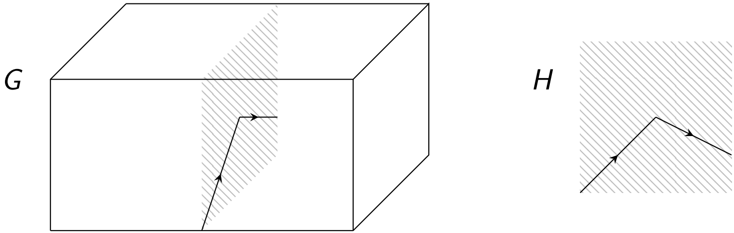


Rank reduction

- A length-minimizing curve contained in a subgroup $H < G$ is also length-minimizing in H.
- A corner is contained in the rank 2 subgroup generated by the 2 half-lines.



Reduction of the regularity problem to Carnot groups

Existence of a length-minimizing curve with a corner-type singularity in some subriemannian manifold.

$$\Longrightarrow$$

Existence of a length-minimizing corner in a rank 2 Carnot group.

Cutting corners in Carnot groups

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Proof by induction on the step of the group.

1. Lift a geodesic from the previous step.
2. Correct the error in the endpoint using the stratification of the Lie algebra.
3. Use dilations to find a situation where there is a decrease of length.

The setting

- A rank 2 Carnot group G of step s, with stratified algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$.
- Linearly independent vectors $X_1, X_2 \in V_1$ of unit norm $|X_1| = |X_2| = 1$.
- A corner

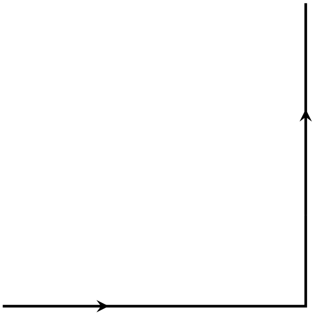
$$t \mapsto \begin{cases} \exp(-tX_1), & t \leq 0 \\ \exp(tX_2), & t > 0 \end{cases}$$

  connecting $\exp(X_1)$ to $\exp(X_2)$ with length 2.
- Need to show that

$$d(\exp(X_1), \exp(X_2)) < 2.$$

The Euclidean case (step 1)

In the Euclidean case, there are only horizontal components:



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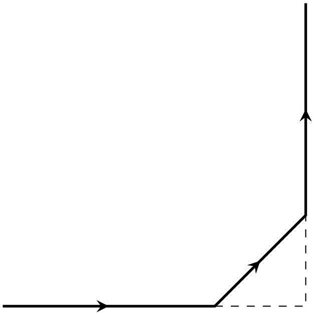
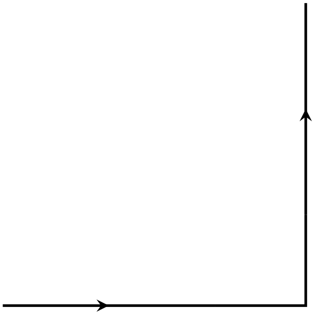


The Heisenberg group case (step 2)

In the Heisenberg case, the error in the vertical component needs to be corrected.



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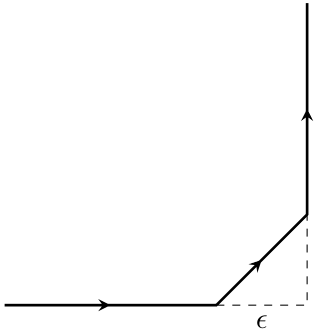

$z_{\text{error}} = \frac{1}{2}\epsilon^2$

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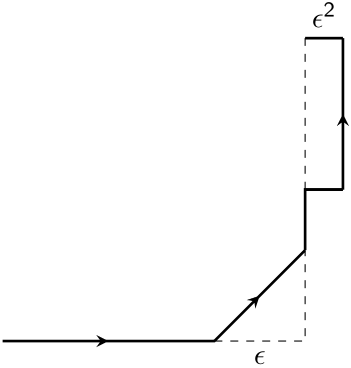

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Lifting curves from step $s-1$ to step s

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Lemma

If corners are not length-minimizing in step $s-1$, then there exists $h \in H$ such that

$$d(\exp(X_1), h \exp(X_2)) < 2.$$

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If corners are not length-minimizing in step $s-1$, then there exists $h \in H$ such that

$$d(\exp(X_1), h\exp(X_2)) < 2.$$

However, in general

$$d(\exp(X_1), h\exp(X_2)) + d(h\exp(X_2), \exp(X_2)) > 2.$$

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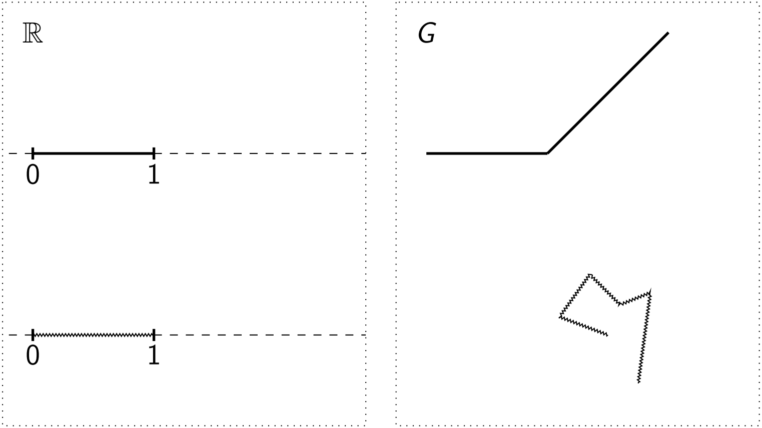

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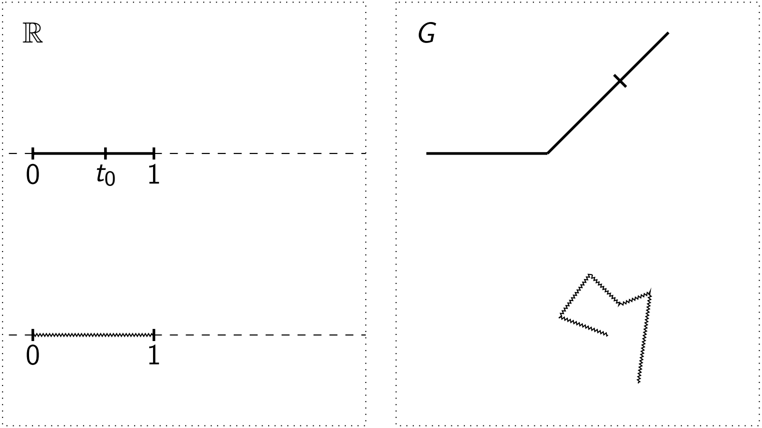

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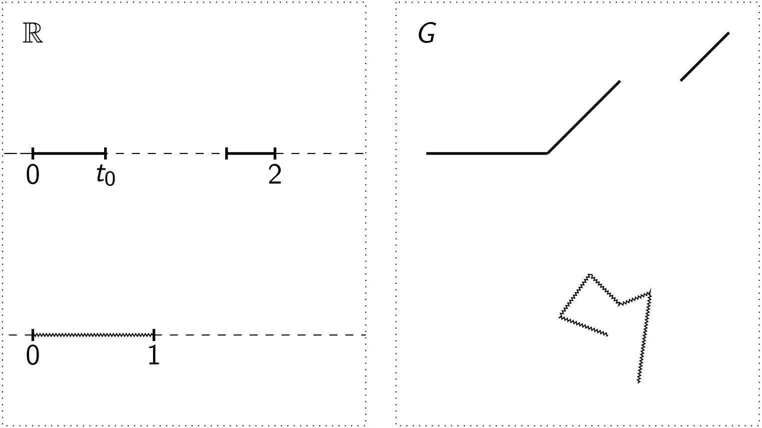

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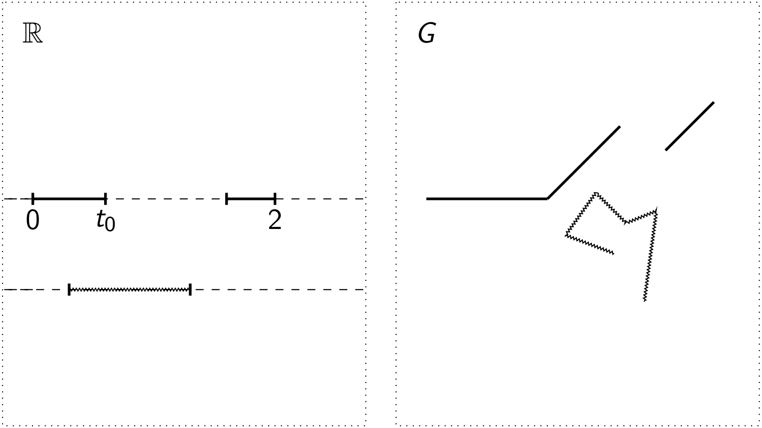

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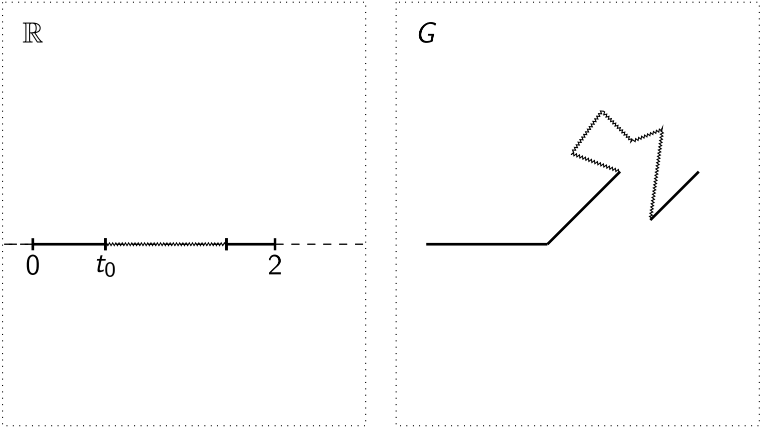

Curve insertions (algebraic viewpoint)

Let α and β be curves $[0,1] \to G$ with $\beta(0) = e$.
The insertion of β into α at time t_0 is the curve $[0,2] \to G$:

$$t \mapsto \begin{cases} \alpha(t), & t < t_0 \\ \alpha(t_0) \cdot \beta(t - t_0), & t_0 < t < t_0 + 1 \\ \alpha(t_0) \cdot \beta(1) \cdot \alpha(t_0)^{-1} \cdot \alpha(t-1), & t_0 + 1 < t \end{cases}$$

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The new endpoint is given by the conjugation

$$\mathsf{C}_{\alpha(t_0)}\, \beta(1) \cdot \alpha(1).$$

Using the stratification

- Insert curves with endpoints in $\exp(V_{s-1})$ along the corner.
- Step $s \geq 3 \implies \exp : V_{s-1} \oplus V_s \to G$ is an injective homomorphism.
- For $X \in V_1$ and $Y \in V_{s-1}$

$$\mathrm{C}_{\exp(X)} \exp(Y) = \exp(Y + [X, Y]).$$

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$\implies$ Corrections have the following **linear** effect:

Insertion point	endpoint	change in layer $s-1$	change in layer s
X	Y	Y	$[X, Y]$

Using the stratification

- The error to be corrected is $h = \exp(Z)$, with $Z \in V_s$.
- G is a Carnot group, so $V_s = [V_1, V_{s-1}]$.
- X_1 and X_2 are linearly independent, so they span V_1.

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$\Longrightarrow$ There exist $W_1, W_2 \in V_{s-1}$ such that

$$Z = [X_1, W_1] + [X_2, W_2] .$$

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Need to choose vectors $Y \in V_{s-1}$ and insertion points $X \in V_1$ along the corner such that

1. The error in the last layer is corrected:

$$\sum [X, Y] = -Z = -[X_1, W_1] - [X_2, W_2]$$

2. No error in the second-to-last layer is created:

$$\sum Y = 0.$$

The error correcting system

Insertion point	endpoint	change in layer $s-1$	change in layer s
X_1	Y_1	Y_1	$[X_1, Y_1]$
$\frac{1}{2}X_2$	Y_2	Y_2	$\left[\frac{1}{2}X_2, Y_2\right]$
X_2	Y_3	Y_3	$[X_2, Y_3]$

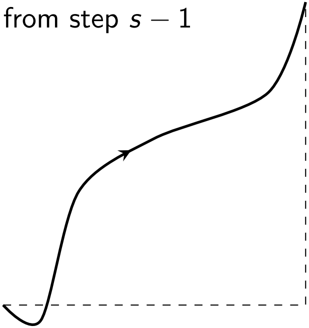
Suffices to solve the linear system

$$\begin{aligned} Y_1 + Y_2 + Y_3 &= 0 \\ [X_1, Y_1] &= -[X_1, W_1] \\ \left[X_2, \tfrac{1}{2}Y_2 + Y_3\right] &= -[X_2, W_2] \end{aligned} \quad \rightarrow \quad \begin{aligned} Y_1 + Y_2 + Y_3 &= 0 \\ Y_1 &= -W_1 \\ \tfrac{1}{2}Y_2 + Y_3 &= -W_2 \end{aligned}$$

Error correction

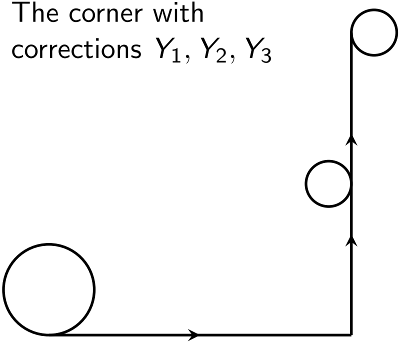
Two different curves with opposite errors in $\exp(V_s)$:

The lift of a geodesic from step $s-1$



Error $\exp(Z)$
Length $2 - C$

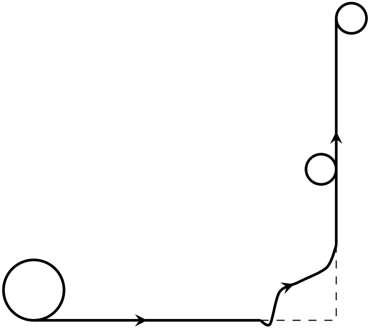
The corner with corrections Y_1, Y_2, Y_3



Error $\exp(-Z)$
Length $2 + \tilde{C}$

Error correction

Cut a small segment of the corner using a dilation of the geodesic from step $s - 1$, and correct the created error by dilating the corrections Y.



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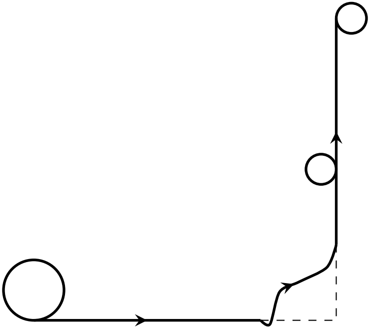

Need to ensure that the scaled versions of the curves still have opposite errors.

Choosing the suitable dilations

- Y_1, Y_2, Y_3 were given by a system that depends linearly on Z.
  $\implies Y_1, Y_2, Y_3$ and Z need to have equal scaling.

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$$\delta_\lambda \exp(Z) = \exp(\lambda^s Z) \quad \text{and}$$
$$\delta_\lambda \exp(Y_j) = \exp(\lambda^{s-1} Y_j)$$

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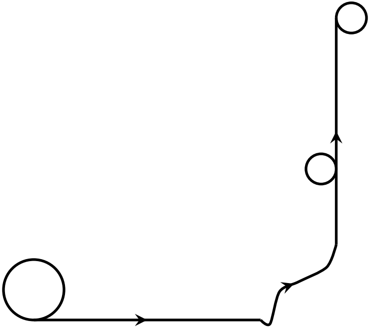
$$\delta_\lambda \exp(Z) = \exp(\lambda^s Z) \quad \text{and}$$
$$\delta_\lambda \exp(Y_j) = \exp(\lambda^{s-1} Y_j)$$

Thus if the geodesic creating the error Z is dilated by ϵ, the corrections Y_1, Y_2, Y_3 need to be dilated by $\epsilon^{s/(s-1)}$:

$$\delta_\epsilon \exp(Z) = \exp(\epsilon^s Z)$$
$$\delta_{\epsilon^{s/(s-1)}} \exp(Y) = \exp(\epsilon^s Y)$$

Choosing the suitable dilations



The length of the combined curve is

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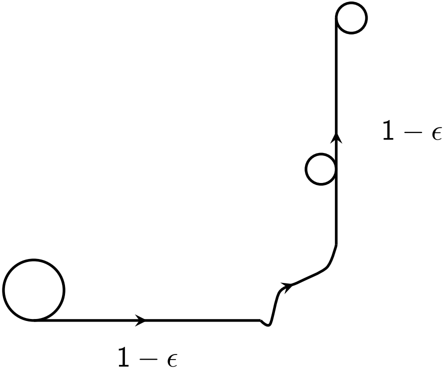

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$$2(1-\epsilon)$$

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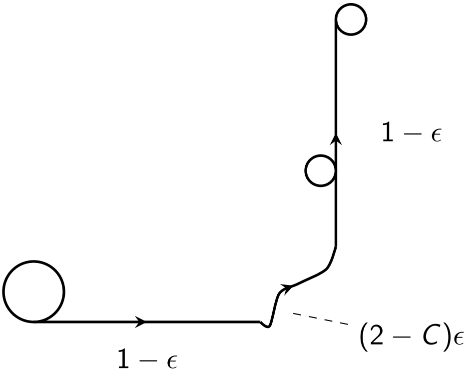

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$$2(1-\epsilon)+(2-C)\epsilon$$

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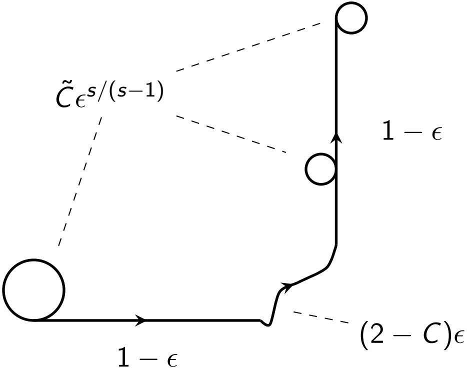

The length of the combined curve is

$$2(1-\epsilon) + (2-C)\epsilon + \tilde{C}\epsilon^{s/(s-1)}$$

Non-minimality of the corner

Simplifying, we get

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Hence

$$d(\exp(X_1),\exp(X_2))\leq 2-C\epsilon+o(\epsilon)<2.$$

for $\epsilon>0$ small enough.

$\Longrightarrow$ A corner from $\exp(X_1)$ to $\exp(X_2)$ is not length-minimizing.