Extendability of sub-Riemannian geodesics

Eero Hakavuori

University of Helsinki

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based on joint works with Andrei Ardentov and Enrico Le Donne

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Chow 1939, Rashevsky 1938: bracket-generating implies induced length metric on M is finite.

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Problem: The subset of horizontal curves connecting x to y can have singularities.

"Abnormal" geodesics

Theorem (Montgomery 1994)

There exists

- *a 3-manifold* M
- *a rank 2 subbundle* $\Delta \subset TM$
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In the C^1-topology, γ is isolated among horizontal curves with the same endpoints.

Normal and abnormal geodesics

Moral: 2 types of sub-Riemannian geodesics

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Caveat: Singularities are easy to detect. Optimality is not.

Visualizing horizontal curves

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Can visualize γ through its *horizontal projection*

$$x = (x_1, \ldots, x_r)\colon [0,1] \to \mathbb{R}^r, \quad \dot x_i(t) = u_i(t).$$

Locally optimal extensions

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- If the sub-Riemannian manifold is complete, then solutions to the geodesic equation exist for all time.

$\implies$ exists a unique locally optimal extension defined on all of $\mathbb{R}$.

Abnormal curves

2. *Abnormal geodesics*: singularities of the family of horizontal curves with fixed endpoints

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Theorem

For every polynomial $P \in \mathbb{R}[x_1, \dots, x_r]$, there exists a sub-Riemannian structure of rank r such that $P(x(t)) \equiv 0 \implies \gamma$ is abnormal.

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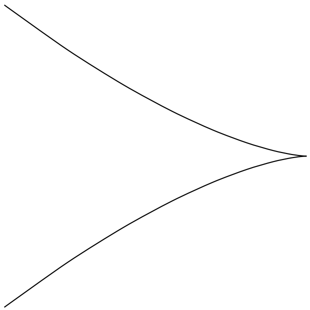
Theorem

For every polynomial vector field P in $\mathbb{R}^r$, there exists a sub-Riemannian structure of rank r such that $\dot x = P(x) \implies \gamma$ is abnormal.

Abnormal curves: non-existent local extensions

Consider the cuspidal cubic $P(x, y) = x^3 + y^2 = 0$.



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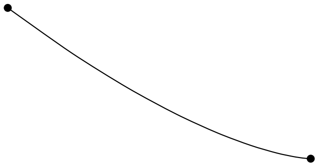
$\rightsquigarrow$ rank 2 sub-Riemannian structure of dimension 8 with abnormal curve:



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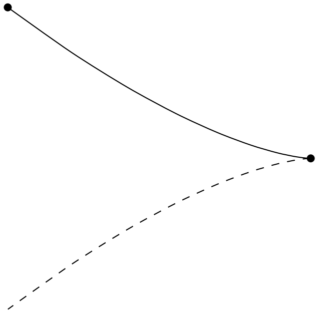
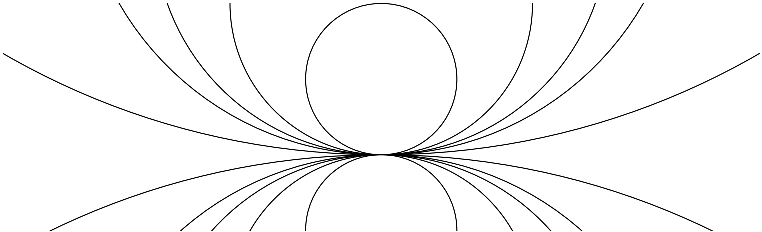
Only possible abnormal extensions contained in $P = 0$.



$\implies$ no locally geodesic extension through 0.

Abnormal curves: non-unique local extensions

Consider the complex differential equation $\dot{z} = z^2$.



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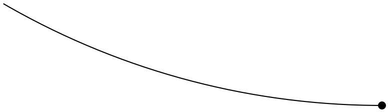
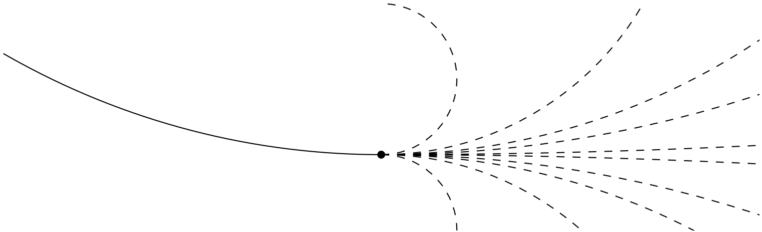
Consider the complex differential equation $\dot{z} = z^2$.
$\rightsquigarrow$ rank 2 sub-Riemannian structure of dimension 1377 with abnormal curve



Abnormal curves: non-unique local extensions

Consider the complex differential equation $\dot{z} = z^2$.
$\rightsquigarrow$ rank 2 sub-Riemannian structure of dimension 1377 with abnormal curve that has non-unique (potentially minimizing) abnormal extensions:



Globally optimal extensions

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Geodesic equation $\implies$ a normal geodesic $\gamma\colon [0,1] \to M$ can be extended locally optimally to $\gamma\colon [0,\infty) \to M$.

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The method of *complete optimal synthesis*:

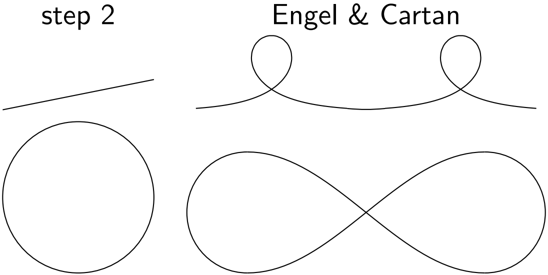
1. Handle all abnormal geodesics separately.
2. Solve the geodesic equation to parametrize all normal geodesics $\rightsquigarrow$ sub-Riemannian exponential map.
3. Use symmetries to find a candidate for the cut time.
4. Prove that the exponential map restricts to a diffeomorphism.

Heisenberg, Engel, and Cartan group geodesics

Optimal synthesis examples:

- Step 2 Carnot groups: Montanari-Morbidelli 2017
- Engel: Ardentov and Sachkov 2015
- Cartan: Sachkov 2003, Sachkov 2021, Ardentov-H. 2022

Horizontal projections of geodesic equation trajectories are periodic:



Cut-times vary between half a period and 1.5 periods.

Infinite optimality

Existence of an infinite extension $\gamma\colon [0,\infty) \to M$ is restrictive.

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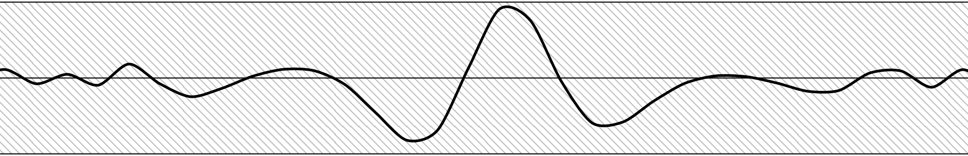
Theorem (H.-Le Donne 2023)

Let M be a rank r sub-Riemannian Carnot group. If $\gamma\colon [0,\infty) \to M$ is a (constant speed) geodesic, there exists a hyperplane W and $R > 0$ such that the horizontal projection $x\colon [0,\infty) \to \mathbb{R}^r$ is contained in a R-neighborhood of W.



Thank you for your attention!