

# Carnot groups and abnormal dynamics

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November 3, 2020

# Sub-Riemannian manifolds

A sub-Riemannian manifold consists of

- a smooth manifold  $M$
- a bracket-generating distribution  $\Delta \subset TM$
- a smoothly varying inner product on  $\Delta$

Assume (for simplicity):

- $\Delta$  has a global frame  $X_1, \dots, X_r$
- the vector fields  $X_1, \dots, X_r$  are complete

# The endpoint map

Fix a base point  $p \in M$ .

## Definition (Endpoint map)

The *endpoint map* is the map

$$\text{End}: L^2([0, 1]; \mathbb{R}^r) \rightarrow M, \quad u \mapsto \gamma_u(1),$$

where  $\gamma_u: [0, 1] \rightarrow M$  is the curve

$$\begin{aligned} \dot{\gamma}_u(t) &= \sum u_i(t) X_i(\gamma_u(t)) \\ \gamma_u(0) &= p \end{aligned}$$

Assumptions  $\implies$  endpoint map well defined and surjective.

# The endpoint map

Abnormal  $\leftrightarrow$  critical points and values of the endpoint map.

**Abnormal control** = critical point  $u \in L^2$  of the endpoint map

**Abnormal curve** = integral curve  $\gamma_u$  of an abnormal control  $u$

**Abnormal set** = the set of critical values of the endpoint map  
= the subset of  $M$  that can be reached from the basepoint with an abnormal curve.

# Open problems

## Conjecture (Sard)

*The abnormal set has zero measure.*

## Conjecture (Regularity)

*All length-minimizing curves are smooth.*

There are two types of length-minimizing curves.

- 1 normal: satisfy a geodesic equation  $\implies$  are smooth
- 2 abnormal: unknown regularity

# A dynamical approach

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

*In sub-Riemannian manifolds of rank 2 and step 4, abnormal minimizers have  $C^1$  regularity.*

Theorem (Boarotto and Vittone 2020)

*In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set of codimension at least one.*

rank = rank of the distribution  $\Delta$

step = length of Lie brackets needed to span  $TM$

step 1	step 2	step 3	step 4
$X_k$	$[X_j, X_k]$	$[X_i, [X_j, X_k]]$	$[X_h, [X_i, [X_j, X_k]]]$

# The homogeneous setting

- $M = G$  is a Carnot group: a nilpotent Lie group whose Lie algebra is stratified

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \dots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}$$

- The basepoint  $p$  is the identity element  $e$ .
- $\Delta$  is the left-invariant distribution with  $\Delta_e = \mathfrak{g}^{[1]}$ .
- $X_1, \dots, X_r$  are left-invariant.

# Characterization of abnormal curves

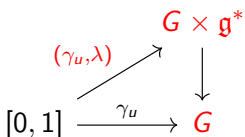
$$\begin{array}{ccc} & & T^*M \\ & \nearrow \lambda & \downarrow \\ [0, 1] & \xrightarrow{\gamma_u} & M \end{array}$$

$\gamma_u: [0, 1] \rightarrow M$  abnormal  $\iff \lambda$  is a characteristic curve of the symplectic form restricted to  $\Delta^\perp$



# Characterization of abnormal curves

$T^*G \simeq G \times \mathfrak{g}^*$  by right-trivialization



$\gamma_u: [0, 1] \rightarrow M$  abnormal  $\iff \lambda \in \mathfrak{g}^*$  constant with  
 $\lambda(\text{Ad}_{\gamma_u(t)} \mathfrak{g}^{[1]}) = 0$

$$\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g}), \quad \text{Ad}_\gamma X = \frac{d}{ds} \gamma \cdot \exp(sX) \cdot \gamma^{-1} \Big|_{s=0}$$

# A dynamical approach

For  $X \in \mathfrak{g}^{[1]}$ , define the *abnormal polynomial*

$$P_X: G \rightarrow \mathbb{R}, \quad P_X(g) = \lambda(\text{Ad}_g X)$$

- $\gamma$  abnormal  $\iff P_X(\gamma(t)) = 0$  for all  $X \in \mathfrak{g}^{[1]}$ .

Idea: consider the (singular) foliation tangent to  $\Delta \cap T\{P_X = 0\}$ .

# A dynamical approach

Rank 2: for  $P = P_X$

$$0 = \frac{d}{dt}P(\gamma_u(t)) = u_1(t)X_1P(\gamma_u(t)) + u_2(t)X_2P(\gamma_u(t)).$$

When  $(X_1P, X_2P) \neq 0$ , up to reparametrization

$$u_1(t) = -X_2P(\gamma_u(t))$$

$$u_2(t) = X_1P(\gamma_u(t))$$

$\implies$  ODE for  $\gamma_u$ .

# A dynamical approach

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

*In sub-Riemannian manifolds of rank 2 and step 4, abnormal minimizers have  $C^1$  regularity.*

Theorem (Boarotto and Vittone 2020)

*In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set of codimension at least one.*

Proof strategy:

- 1 The dynamics is linear.
- 2 Separate cases by the Jordan form of the linear part.
- 3 Study the dynamics explicitly in the normal forms.

# Abnormal dynamics is complicated

## Theorem (H. 2020)

Let  $\dot{x} = P(x)$  be a polynomial ODE system in  $\mathbb{R}^r$ .

There exists a Carnot group of rank  $r$  such that all trajectories of the ODE lift to abnormal curves.

For  $x = (x_1, \dots, x_r)$ , a lift is  $\gamma_u$  where  $u_i = \dot{x}_i$ .

Proof idea:

- 1 Every polynomial ODE has a polynomial first integral in a lift.
- 2 Curves contained in an algebraic variety are abnormal in a lift.

# Construction of a first integral

## Theorem (H. 2020)

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# Horizontal gradients

## Lemma

*Every polynomial vector field  $P: \mathbb{R}^r \rightarrow \mathbb{R}^r$  is the horizontal gradient of some polynomial in a Carnot group of high enough step.*

For the frame  $X_1, \dots, X_r$  the horizontal gradient of  $Q: G \rightarrow \mathbb{R}$  is

$$\nabla_{\text{hor}} Q = \sum (X_i Q) X_i: G \rightarrow TG.$$

In coordinates, lift  $P: \mathbb{R}^r \rightarrow \mathbb{R}^r$  to the horizontal vector field

$$P: G \rightarrow TG, \quad P(x_1, \dots, x_r, \dots, x_n) = \sum_{i=1}^r P_i(x_1, \dots, x_r) X_i(x)$$

# Gradients in $\mathbb{R}^r$

$$P = (P_1, \dots, P_r) = \nabla Q \text{ for some } Q: \mathbb{R}^r \rightarrow \mathbb{R} \iff \partial_i P_j = \partial_j P_i$$

Recursion for  $Q$ :

$$Q_1 = \int P_1 dx_1$$

$$Q_2 = Q_1 + \int (P_2 - \partial_2 Q_1) dx_2$$

$$\vdots$$

$$Q = Q_r = Q_{r-1} + \int (P_r - \partial_r Q_{r-1}) dx_r$$



# A non-gradient vector field in $\mathbb{R}^r$

$P(x) = (x_1 - x_2, x_1 + x_2) \neq \nabla Q$  for any  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Lift to a horizontal vector field in the Heisenberg group.

$$X_1(x) = \partial_1$$

$$X_2(x) = \partial_2 + x_1 \partial_3$$

$$X_3(x) = [X_1, X_2](x) = \partial_3$$

$$P: H \rightarrow TH, \quad P(x) = (x_1 - x_2)X_1(x) + (x_1 + x_2)X_2(x)$$

Then  $P = \nabla_{\text{hor}} Q$  for the polynomial

$$Q(x) = \frac{1}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 + 2x_3$$

# Recursion for horizontal gradient integration

$$X_1 Q = x_1 - x_2$$

$$X_2 Q = x_1 + x_2$$

Compute commutators:

$$X_3 Q = [X_1, X_2] Q = X_1(X_2 Q) - X_2(X_1 Q) = 2$$

Integrate backwards:

$$Q_3 = \int X_3 Q dx_3$$

$$Q_2 = Q_3 + \int (X_2 Q - X_2 Q_3) dx_2$$

$$\begin{aligned} Q = Q_1 &= Q_2 + \int (X_1 Q - X_1 Q_2) dx_1 \\ &= \frac{1}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 + 2x_3 \end{aligned}$$

# Recursion for horizontal gradient integration

Why it works:

- As *weighted* differential operators,  $[X_1, X_2]$  is a degree 2 operator,  $[X_1, [X_1, X_2]]$  is degree 3, etc.  
     $\implies$  partial derivatives of a polynomial eventually vanish
- There exist coordinates such that  $X_i = \partial_i + \sum_{j>i} c_{ij} \partial_j$ .  
     $\implies$  integration variable by variable is possible

# A horizontal first integral

For an ODE

$$\dot{x}_i = P_i(x), \quad x \in \mathbb{R}^r, \quad i = 1, \dots, n$$

integrate any nonzero orthogonal vector field.

E.g. if  $P_1 \neq 0$ , integrate

$$X_1 Q = -P_2, \quad X_2 Q = P_1 \quad X_3 Q = X_4 Q = \dots = X_r Q = 0.$$

Then for a trajectory  $x: [0, 1] \rightarrow G$  of  $\dot{x} = \sum P_i(x) X_i(x)$

$$\frac{d}{dt} Q(x) = P_1(x) X_1 Q(x) + \dots + P_r(x) X_r Q(x) = 0.$$

# Abnormal factors

## Theorem (H. 2020)

*Let  $\dot{x} = P(x)$  be a polynomial ODE system in  $\mathbb{R}^r$ .*

*There exists a Carnot group of rank  $r$  such that all trajectories of the ODE lift to abnormal curves.*

Proof idea:

- 1 Every polynomial ODE has a polynomial first integral in a lift.
- 2 Curves contained in an algebraic variety are abnormal in a lift.

# Higher order abnormality

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \dots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}.$$

## Definition

$$\gamma: [0, 1] \rightarrow G \text{ abnormal} \iff \lambda(\text{Ad}_{\gamma(t)} \mathfrak{g}^{[1]}) = 0$$

## Definition

$$\gamma \text{ abnormal of order } k \iff \lambda(\text{Ad}_{\gamma(t)}(\mathfrak{g}^{[1]} \oplus \dots \oplus \mathfrak{g}^{[k]})) = 0$$

## Lemma

*If  $\gamma(0) = e$  and  $\lambda(\text{Ad}_{\gamma(t)} \mathfrak{g}^{[k]}) = 0$ , then  $\gamma$  is abnormal of order  $k$ .*

# Abnormal factors

## Proposition

*For any polynomial  $Q: H \rightarrow \mathbb{R}$ , there exists*

- *a Carnot group  $G$  with a projection  $\pi: G \rightarrow H$*
- *$\lambda \in \mathfrak{g}^*$*
- *$k \in \mathbb{N}$*

*such that  $Q \circ \pi: G \rightarrow \mathbb{R}$  is a factor of the polynomial  $x \mapsto \lambda(\text{Ad}_x Y)$  for every  $Y \in \mathfrak{g}^{[k]}$ .*

# Abnormal factors proof

Consider a linear system

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \dots, m$$

in the variables  $(\lambda, \nu)$ , where

- $P_i^\lambda(x) = \lambda(\text{Ad}_x Y_i)$  for a basis  $Y_1, \dots, Y_m$  of  $\mathfrak{g}^{[k]}$
- $S_i^\nu$  are generic polynomials of the form

$$S^\nu = \nu_0 + \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 + \nu_4 x_1^2 + \nu_5 x_1 x_2 + \nu_6 x_2^2 + \dots$$

such that  $\deg(S_i^\nu) + \deg(Q) = \deg(P_i)$ .



# Abnormal factors proof

Let

- $k = \deg Q + 1$
- $G_s$  a free Carnot group of step  $s$

## Lemma

*The linear system*

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \dots, m$$

*has a non-trivial solution  $(\lambda, \nu)$  in  $G_s$  for large  $s$ .*

# Monomial counting

Proof of Lemma:

- ① Hall basis argument  $\implies \exists \lambda = \lambda(\nu)$  such that  $P_1^{\lambda(\nu)} = Q \cdot S_1^\nu$   
Consider the remaining system

$$P_i^{\lambda(\nu)} = Q \cdot S_i^\nu, \quad i = 2, \dots, m$$

- ② In step  $s$ ,  $\deg(P_i^\lambda) \leq s - k$ . The number of equations is

$$(m - 1) \cdot \#\{\text{monomials of degree up to } s - k\}$$

and the number of variables is

$$m \cdot \#\{\text{monomials of degree up to } s - k - \deg(Q)\}$$

- ③ Poincaré series asymptotics for  $s \rightarrow \infty$   
 $\implies \#\text{variables} \gg \#\text{equations}.$

# The entire proof

## Theorem (H. 2020)

Let  $\dot{x} = P(x)$  be a polynomial ODE system in  $\mathbb{R}^r$ .

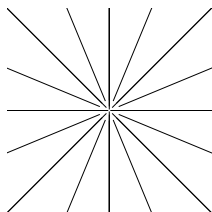
*There exists a Carnot group of rank  $r$  such that all trajectories of the ODE lift to abnormal curves.*

Proof:

- 1 Every polynomial ODE has a polynomial first integral in a lift.
  - Consider an orthogonal vector field.
  - Every polynomial vector field is a horizontal gradient.
- 2 Curves contained in an algebraic variety are abnormal in a lift.
  - Common factors of abnormal polynomials = linear system.
  - Monomial counting  $\implies$  the system is underdetermined.

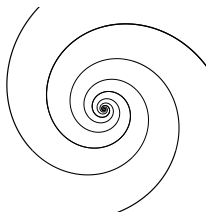
# Linear ODEs

Abnormals in the free Carnot group of rank 2 and step 7



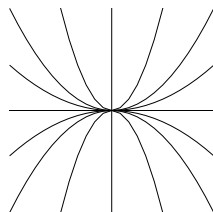
$$\dot{x} = x$$

$$\dot{y} = y$$



$$\dot{x} = -\frac{1}{4}x - y$$

$$\dot{y} = x - \frac{1}{4}y$$



$$\dot{x} = x$$

$$\dot{y} = 2y$$

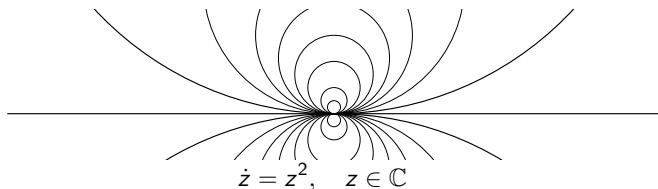
$\exists \lambda: \mathbb{R}^6 \rightarrow \mathfrak{g}^*$  semi-algebraic such that trajectories of

$$\dot{x} = ax + by + c \quad \dot{y} = dx + ey + f$$

are abnormal with covector  $\lambda(a, b, c, d, e, f)$ .

# Concatenations of trajectories

Abnormals in the free Carnot group of rank 2 and step 13



Let  $E \subset [0, 1]$  be nowhere dense.  $\exists$  abnormal curve that is

- injective
- parametrized by arc length on  $[0, 1] \setminus E$
- not  $C^2$  at any point  $x \in E$
- if  $E$  is perfect, not  $C^1$  at any point  $x \in E$

# An inefficient formula

Let  $P: \mathbb{R}^r \rightarrow \mathbb{R}^r$  be a polynomial vector field.

Let  $d(r, k) = \dim \mathfrak{f}_r^{[k]} = \frac{1}{k} \sum_{d|k} \mu(d) r^{k/d}$ .

Consider the rational function

$$\sum_{k=0}^{\infty} C_k t^k = \frac{\left(1 - (d(r, \deg(P) + 1))(1 - t^{\deg(P)})\right) t^{\deg(P)+1}}{\prod_{k=1}^{\deg(P)} (1 - t^k)^{d(r, k)}}.$$

If  $\sum_{k=0}^s C_k > 0$ , then trajectories of  $P$  are abnormal in step  $s$ .

# Inefficient numbers from an inefficient formula

$P$  a polynomial vector field in  $\mathbb{R}^r$ . Trajectories abnormal in step:

$r \backslash \deg(P)$	1	2	3	4	5
2	11	38	172	577	2372
3	89	724	6034	46036	365813
4	386	5322	73109	983505	13529000

## Example

A polynomial vector field in  $\mathbb{R}^4$  of degree 5 has abnormal lifts in the free Carnot group  $G$  of rank 4 and step 13529000.

$$\dim G \approx 4.1338 \cdot 10^{8145262}$$

# Inefficient numbers from an inefficient formula

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## Example

A polynomial vector field in  $\mathbb{R}^4$  of degree 5 has abnormal lifts in the free Carnot group  $G$  of rank 4 and step **2372**?

$$\dim G \approx 6.857 \cdot 10^{1425}$$

## Conjecture

*The abnormality step only depends on  $\deg(P)$  and not the rank  $r$ .*



Thank you for your attention!