Sub-Riemannian abnormal extremals

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Sub-Riemannian manifolds

A sub-Riemannian manifold consists of

- a smooth manifold M
- a bracket-generating distribution $\Delta \subset TM$
- a smoothly varying inner product on Δ

Assume (for simplicity):

- Δ has a global orthonormal frame $X_1, \ldots, X_r$
- the vector fields $X_1, \ldots, X_r$ are complete

The endpoint map

Fix a base point $p \in M$.

Definition (Endpoint map)

The *endpoint map* is the map

$$\mathrm{End}\colon L^2([0,1];\mathbb{R}^r) \to M, \quad u \mapsto \gamma_u(1),$$

where $\gamma_u\colon [0,1] \to M$ is the curve

$$\dot\gamma_u(t) = \sum u_i(t) X_i(\gamma_u(t))$$
$$\gamma_u(0) = p$$

Assumptions $\implies$ endpoint map well defined and surjective.

The endpoint map

Abnormal $\leftrightarrow$ critical points and values of the endpoint map.

Abnormal control $=$ critical point $u \in L^2$ of the endpoint map

Abnormal curve $=$ integral curve γ_u of an abnormal control u

Abnormal set $=$ the set of critical values of the endpoint map

The endpoint map

Abnormal $\leftrightarrow$ critical points and values of the endpoint map.

$$\begin{aligned}
\text{Abnormal control} &= \text{critical point } u \in L^2 \text{ of the endpoint map} \\
\text{Abnormal curve} &= \text{integral curve } \gamma_u \text{ of an abnormal control } u \\
\text{Abnormal set} &= \text{the set of critical values of the endpoint map} \\
&= \text{the subset of } M \text{ that can be reached from the basepoint with an abnormal curve.}
\end{aligned}$$

Open problems

Conjecture (Sard)

The abnormal set has zero measure.

Conjecture (Regularity)

All length-minimizing curves are smooth.

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All length-minimizing curves are smooth.

The two types of length-minimizing curves.

1. normal: satisfy a geodesic equation $\implies$ are smooth
2. abnormal: unknown regularity

Some regularity results

- Strichartz 1986: C^∞-regularity for strongly bracket generating structures
- H. and Le Donne 2016: geodesics do not have corner-type singularities
- Monti, Pigati, and Vittone 2018: existence of tangent lines
- Belotto da Silva, Figalli, Parusiński, and Rifford 2018: C^1-regularity for 3-dimensional analytic sub-Riemannian manifolds
- Barilari, Chitour, Jean, Prandi, and Sigalotti 2020: C^1-regularity for rank 2 step 4 sub-Riemannian structures

Some Sard results

Assume the sub-Riemannian structure is analytic.
Then the abnormal set is ...

- ...contained in a closed nowhere dense set (Agrachëv 2009)
- ...a countable union of semianalytic curves in the case of 3d-manifolds (Belotto da Silva, Figalli, Parusiński, and Rifford 2018)
- ...a proper algebraic subvariety in Carnot groups of step 2, in $\mathbb{F}_{2,4}$ (free Carnot group of rank 2 step 4), and in $\mathbb{F}_{3,3}$ (Le Donne, Montgomery, Ottazzi, Pansu, and Vittone 2016)
- ...a proper sub-analytic subvariety in Carnot groups of rank 3 step 3, and in rank 2 step 4 (Boarotto and Vittone 2020)

Carnot groups

- a Carnot group G: a nilpotent Lie group whose Lie algebra is stratified

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}$$

- The basepoint p is the identity element e.
- Δ is the left-invariant distribution with $\Delta_e = \mathfrak{g}^{[1]}$.
- The orthonormal frame $X_1, \ldots, X_r$ is left-invariant.

Part I: The metric approach

Gromov-Hausdorff convergence

- (Z, d) a metric space
- $X_1, X_2 \subset Z$

Definition (Hausdorff distance)

$$d_H(X_1, X_2) = \inf\{r > 0 : X_1 \subset B(X_2, r) \text{ and } X_2 \subset B(X_1, r)\}$$

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- (X_1, d_1), (X_2, d_2) metric spaces
- (Z, d) metric space such that $(X_1, d_1) \hookrightarrow (Z, d)$ and $(X_2, d_2) \hookrightarrow (Z, d)$ isometrically.

Definition (Gromov-Hausdorff distance)

$$d_{GH}(X_1, X_2) = \inf_Z d_H(X_1, X_2)$$

Gromov-Hausdorff convergence

- (X_k, x_k, d_k), $k \in \mathbb{N}$, pointed metric spaces

Definition (Pointed Gromov-Hausdorff convergence)

$(X_k, x_k, d_k) \xrightarrow{GH} (Y, y, d_Y)$ if $\quad \forall r \quad \forall \epsilon \quad \exists k_0 \quad \forall k > k_0$
$\exists$ *Gromov-Hausdorff approximation* $f : B(x_k, r) \subset X_k \to Y$ with

- f distorts distance by at most ϵ
- f preserves the basepoint
- f is ϵ-almost surjective onto the r ball

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- $|d(f(x), f(z)) - d(x, z)| < \epsilon$
- $f(x_k) = y$
- $B(y, r - \epsilon) \subset B(f(B(x_k, r)), \epsilon)$

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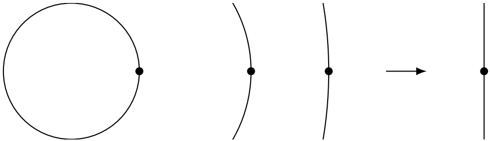
Example: $(S^1(0, r), (r, 0)) \xrightarrow{GH} (\mathbb{R}, 0)$ as $r \to \infty$.



Metric tangents

Definition

(Y, y, d_Y) is a metric tangent to (X, d_X) at $x \in X$ if $(X, x, \lambda d_X) \xrightarrow{GH} (Y, y, d_Y)$ as $\lambda \to \infty$.

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Theorem (Mitchell 1985)

The metric tangent of an equiregular sub-Riemannian manifold is a sub-Riemannian Carnot group.

Theorem (Bellaïche 1996)

The metric tangent of any sub-Riemannian manifold is a sub-Riemannian homogeneous space (=a quotient of a sub-Riemannian Carnot group).

Metric tangents to geodesics

- (M, d) a sub-Riemannian manifold
- $p \in M$ a basepoint
- $\gamma\colon (-1, 1) \to M$ a geodesic through $\gamma(0) = p$
- $(M, p, \lambda d) \xrightarrow{GH} (G, e, d_G)$ as $\lambda \to \infty$
- $f_{\lambda,r,\epsilon}\colon B(p, r/\lambda) \to G$ Gromov-Hausdorff approximations

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Definition

$$\mathrm{Tan}(\gamma, 0) = \{\sigma : (\gamma, \gamma(0), \lambda_k d) \xrightarrow{GH} (\sigma, \sigma(0), d_G), \lambda_k \to \infty\}$$

Metric tangents to geodesics

Immediate consequences:

Lemma

γ *geodesic* $\implies$ *every* $\sigma \in \mathrm{Tan}(\gamma, 0)$ *is a geodesic*

Lemma

$\mathrm{Tan}(\mathrm{Tan}(\gamma, t), 0) \subset \mathrm{Tan}(\gamma, t)$.

Metric tangents to geodesics

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Proof: $f_{\lambda,r,\epsilon}$ are ϵ-quasi-isometries.

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$\mathrm{Tan}(\mathrm{Tan}(\gamma, t), 0) \subset \mathrm{Tan}(\gamma, t)$.

Proof: a diagonal argument.

Metric tangents in Carnot groups

- M sub-Riemannian manifold
- $(M, p, \lambda d) \xrightarrow{GH} (G, e, d_G)$
- $\gamma\colon (-1,1) \to M$, $\gamma(0) = p$
- $\sigma\colon \mathbb{R} \to G$, $\sigma(0) = e$

Gromov-Hausdorff convergence $(\gamma, \gamma(0), \lambda d) \xrightarrow{GH} (\sigma, \sigma(0), d_G)$

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Gromov-Hausdorff convergence $(\gamma, \gamma(0), \lambda d) \xrightarrow{GH} (\sigma, \sigma(0), d_G)$

- G sub-Riemannian Carnot group
- (G, e, d) and $(G, e, \lambda d)$ are isometric by dilation $\delta_\lambda\colon G \to G$ $\implies (G, e, \lambda d) \xrightarrow{GH} (G, e, d)$

$\gamma_\lambda \to \sigma$ uniformly on compact sets, where

$$\gamma_\lambda\colon (-\lambda, \lambda) \to G, \quad \gamma_\lambda(t) = \delta_\lambda(\gamma(t/\lambda)).$$

Metric tangents to geodesics

- $\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}$
- $G = \exp(\mathfrak{g})$ a Carnot group
- $\pi_s \colon G \to G/\exp(\mathfrak{g}^{[s]})$ the quotient projection down one step

Theorem (H. and Le Donne 2018)

$\gamma : (-1,1) \to G$ geodesic and $\sigma \in \mathrm{Tan}(\gamma, 0)$.
Then $\pi_s \circ \sigma \colon \mathbb{R} \to G/\exp(V_s)$ is also a geodesic.

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Corollary

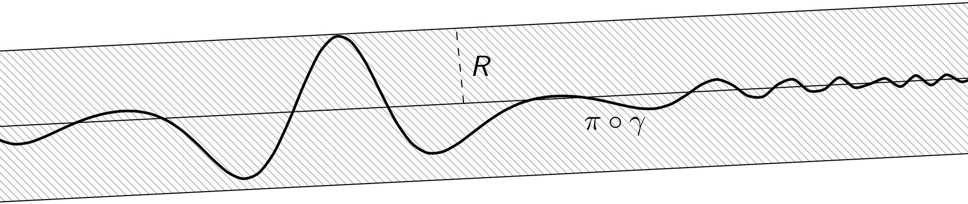
$\gamma : (-1,1) \to G$ geodesic and
$\sigma \in \mathrm{Tan}^s(\gamma, 0) = \mathrm{Tan}(\mathrm{Tan}(\cdots \mathrm{Tan}(\gamma, 0), \cdots, 0), 0)$.
Then $\pi \circ \sigma \colon \mathbb{R} \to \mathbb{R}^{\dim \mathfrak{g}^{[1]}}$ is a geodesic.
That is, $\sigma(t) = \exp(tX)$ for some $X \in \mathfrak{g}^{[1]}$.

Large scale behaviour of geodesics

- G a Carnot group
- $r = \dim \mathfrak{g}^{[1]}$
- $\pi \colon G \to \mathbb{R}^r$ the horizontal projection

Theorem (H. and Le Donne 2018)

$\sigma : \mathbb{R} \to G$ a geodesic.
$\exists$ a hyperplane $W \subset \mathbb{R}^r$ and $\exists R > 0$ such that
$\pi \circ \gamma(\mathbb{R}) \subset B(W, R)$.



The cut & correct method

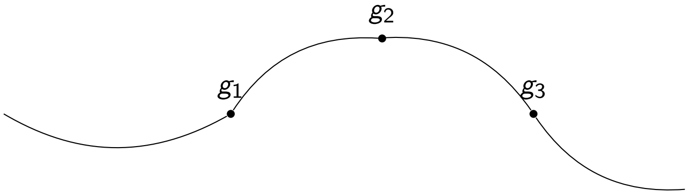
A non-minimality proof strategy (Leonardi and Monti 2008):

1. The cut: replace $\sigma|_{[a,b]}$ with the lift of a geodesic from a lower step Carnot group
2. The correction: perturb the curve so that
   - the endpoint is reverted to the original endpoint, and
   - length remains smaller than the original curve's

The cut & correct method – discretization

- Choose points $g_1, \ldots, g_m$ along the geodesic σ
- Write the endpoint of σ as

$$\sigma(1) = \sigma(0) \cdot \sigma(0)^{-1} \cdot g_1 \cdot g_1^{-1} \cdot g_2 \cdots g_{m-1}^{-1} \cdot g_m \cdot g_m^{-1} \cdot \sigma(1)$$
$$= \sigma(0) \cdot (\sigma(0)^{-1} \cdot g_1) \cdot (g_1^{-1} \cdot g_2) \cdots (g_{m-1}^{-1} \cdot g_m) \cdot (g_m^{-1} \cdot \sigma(1))$$



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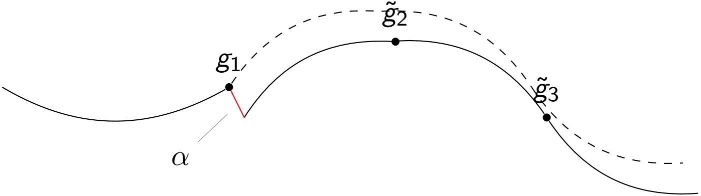
- Easy to insert a perturbation curve $\alpha\colon [0,1] \to G$:

$$\tilde{\sigma}(1) = \sigma(0) \cdot (\sigma(0)^{-1} \cdot g_1) \cdot (\alpha(0)^{-1} \cdot \alpha(1)) \cdot (g_1^{-1} \cdot g_2) \cdots$$

- Perturbed points: $\tilde{g}_k = g_1 \cdot \alpha(0)^{-1} \cdot \alpha(1) \cdot g_1^{-1} \cdot g_k$



The cut & correct method – the cut

Lifting a geodesic from a lower step group in the discretization:

Lemma

$\forall g \in G \quad \exists h \in \exp(\mathfrak{g}^{[s]})$:

$$d_{G/\exp(\mathfrak{g}^{[s]})}(e, \pi_s(g)) = d_G(e, hg).$$

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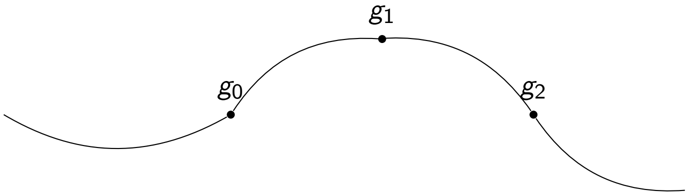
After replacing $\sigma|_{[a,b]}$ with a geodesic segment from $G/\exp(\mathfrak{g}^{[s]})$, either

1. length decreases and the endpoint is translated by $h \in \exp(\mathfrak{g}^{[s]})$, or
2. length does not change, so $\pi_s \circ \sigma|_{[a,b]}$ was already a geodesic

The cut & correct method – the correction

1. Choose $r+1$ points $g_0, \ldots, g_r$ along the curve γ.
2. For each curve segment g_{k-1} to g_k, insert α_k at g_{k-1}, and insert the reverse α_k^{-1} at g_k.



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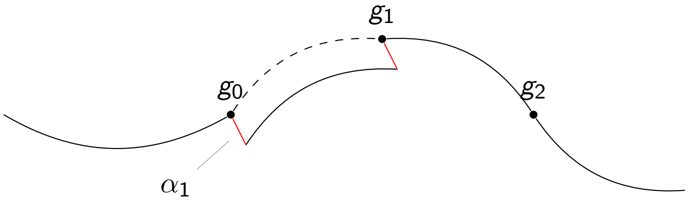

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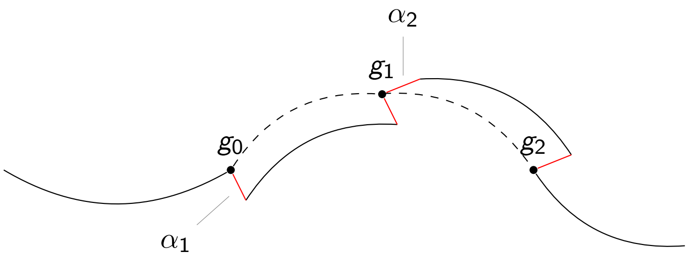

The cut & correct method – the correction

A back-and-forth perturbation is a group commutator:

$$a\alpha a^{-1} \cdot b\alpha^{-1} b = a[\alpha, a^{-1} b]a^{-1}.$$

$\Longrightarrow$ Perturbation in the layer $s-1$ corrects an error in layer s.

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Need to solve

$$L(\alpha_1, \dots, \alpha_r) = \log h \in \mathfrak{g}^{[s]},$$

where $L : (\mathfrak{g}^{[s-1]})^r \to \mathfrak{g}^{[s]}$ is linear.

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Key ingredients:

- bracket-generating $\implies$ L is surjective
- norm of the right-inverse of L is controlled by the horizontal projection of $g_0, \ldots, g_r$

Part II: Abnormal dynamics

Characterization of abnormal curves

Recall

- G a Carnot group of rank r
- $X_1, \dots, X_r$ orthonormal left-invariant frame
- $u \in [0,1] \to \mathbb{R}^r$, $\gamma_u : [0,1] \to G$

  $$\dot\gamma_u(t) = \sum u_i(t) X_i(\gamma_u(t))$$

  $$\gamma_u(0) = p$$
- γ_u abnormal $\iff$ u critical point of $u \mapsto \gamma_u(1)$

Characterization of abnormal curves

$$\begin{array}{ccc} & & T^*M \\ & \overset{\lambda}{\nearrow} & \downarrow \\ [0,1] & \xrightarrow{\gamma_u} & M \end{array}$$

$\gamma_u \colon [0,1] \to M$ abnormal $\iff$ λ is a characteristic curve of the symplectic form restricted to $\Delta^\perp$ (Hsu 1992)

Characterization of abnormal curves

$T^*G \simeq G \times \mathfrak{g}^*$ by right-trivialization

$$\begin{array}{ccc} & & G \times \mathfrak{g}^* \\ & \overset{(\gamma_u,\lambda)}{\nearrow} & \downarrow \\ [0,1] & \xrightarrow{\gamma_u} & G \end{array}$$

$\gamma_u \colon [0,1] \to M$ abnormal $\iff$ $\lambda \in \mathfrak{g}^*$ constant with $\lambda(\mathrm{Ad}_{\gamma_u(t)}\, \mathfrak{g}^{[1]}) = 0$

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}), \quad \mathrm{Ad}_\gamma X = \frac{d}{ds}\, \gamma \cdot \exp(sX) \cdot \gamma^{-1}\Big|_{s=0}$$

Characterization of abnormal curves

For $X \in \mathfrak{g}^{[1]}$, define the *abnormal polynomial*

$$P_X : G \to \mathbb{R}, \quad P_X(g) = \lambda(\mathrm{Ad}_g X)$$

- γ abnormal $\iff$ $P_X(\gamma(t)) = 0$ for all $X \in \mathfrak{g}^{[1]}$.

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Abnormal dynamics: consider the (singular) foliation tangent to $\Delta \cap T\{P_X = 0\}$.

A dynamical approach

Rank 2: for $P = P_X$

$$0 = \frac{d}{dt} P(\gamma_u(t)) = u_1(t) X_1 P(\gamma_u(t)) + u_2(t) X_2 P(\gamma_u(t)).$$

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$$0 = \frac{d}{dt} P(\gamma_u(t)) = u_1(t) X_1 P(\gamma_u(t)) + u_2(t) X_2 P(\gamma_u(t)).$$

When $(X_1 P, X_2 P) \neq 0$, up to reparametrization

$$u_1(t) = -X_2 P(\gamma_u(t))$$
$$u_2(t) = X_1 P(\gamma_u(t))$$

$\implies$ ODE for γ_u.

A dynamical approach

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

In sub-Riemannian manifolds of rank 2 and step 4, abnormal minimizers have C^1 regularity.

Theorem (Boarotto and Vittone 2020)

In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set of codimension at least one.

Proof strategy:

1. The dynamics is linear.
2. Separate cases by the Jordan form of the linear part.
3. Study the dynamics explicitly in the normal forms.

Abnormal dynamics is complicated

Theorem (H. 2020)

Let $\dot{x} = P(x)$ be a polynomial ODE system in $\mathbb{R}^r$.
There exists a Carnot group of rank r such that all trajectories of the ODE lift to abnormal curves.

For $x = (x_1, \ldots, x_r)$, a lift is γ_u where $u_i = \dot{x}_i$.

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There exists a Carnot group of rank r such that all trajectories of the ODE lift to abnormal curves.

For $x = (x_1, \ldots, x_r)$, a lift is γ_u where $u_i = \dot{x}_i$.

Proof idea:

1. Every polynomial ODE has a polynomial first integral in a lift.
2. Curves contained in an algebraic variety are abnormal in a lift.

Construction of a first integral

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In coordinates, lift $P\colon \mathbb{R}^r \to \mathbb{R}^r$ to the horizontal vector field

$$P\colon G \to TG, \quad P(x_1, \ldots, x_r, \ldots, x_n) = \sum_{i=1}^{r} P_i(x_1, \ldots, x_r) X_i(x)$$

Gradients in $\mathbb{R}^r$

$$P = (P_1, \ldots, P_r) = \nabla Q \text{ for some } Q\colon \mathbb{R}^r \to \mathbb{R} \iff \partial_i P_j = \partial_j P_i$$

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Recursion for Q:

$$Q_1 = \int P_1 \, dx_1$$

$$Q_2 = Q_1 + \int (P_2 - \partial_2 Q_1) \, dx_2$$

$$\vdots$$

$$Q = Q_r = Q_{r-1} + \int (P_r - \partial_r Q_{r-1}) \, dx_r$$

A non-gradient vector field in $\mathbb{R}^r$

$$P(x) = (x_1 - x_2, x_1 + x_2) \neq \nabla Q \text{ for any } Q\colon \mathbb{R}^2 \to \mathbb{R}.$$

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Lift to a horizontal vector field in the Heisenberg group.

$$X_1(x) = \partial_1$$
$$X_2(x) = \partial_2 + x_1\partial_3$$
$$X_3(x) = [X_1, X_2](x) = \partial_3$$

$$P\colon H \to TH, \quad P(x) = (x_1 - x_2)X_1(x) + (x_1 + x_2)X_2(x)$$

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$$P \colon H \to TH, \quad P(x) = (x_1 - x_2)X_1(x) + (x_1 + x_2)X_2(x)$$

Then $P = \nabla_{\text{hor}} Q$ for the polynomial

$$Q(x) = \frac{1}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 + 2x_3$$

Recursion for horizontal gradient integration

$$X_1 Q = x_1 - x_2$$
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Integrate backwards:

$$Q_3 = \int X_3 Q \, dx_3$$
$$Q_2 = Q_3 + \int (X_2 Q - X_2 Q_3) \, dx_2$$
$$Q = Q_1 = Q_2 + \int (X_1 Q - X_1 Q_2) \, dx_1$$
$$= \frac{1}{2}x_1^2 - x_1 x_2 + \frac{1}{2}x_2^2 + 2x_3$$

Recursion for horizontal gradient integration

Why it works:

- As *weighted* differential operators, $[X_1, X_2]$ is a degree 2 operator, $[X_1, [X_1, X_2]]$ is degree 3, etc.
  $\implies$ partial derivatives of a polynomial eventually vanish
- There exist coordinates such that $X_i = \partial_i + \sum_{j>i} c_{ij}\partial_j$.
  $\implies$ integration variable by variable is possible

A horizontal first integral

For an ODE

$$\dot{x}_i = P_i(x), \quad x \in \mathbb{R}^r, \quad i = 1, \dots, n$$

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Then for a trajectory $x\colon [0,1] \to G$ of $\dot{x} = \sum P_i(x) X_i(x)$

$$\frac{d}{dt} Q(x) = P_1(x) X_1 Q(x) + \cdots + P_r(x) X_r Q(x) = 0.$$

Abnormal factors

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Higher order abnormality

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}.$$

Definition

$\gamma \colon [0,1] \to G$ abnormal $\iff \lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[1]}) = 0$

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Lemma

If $\gamma(0) = e$ and $\lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[k]}) = 0$, then γ is abnormal of order k.

Abnormal factors

Proposition

For any polynomial $Q\colon H \to \mathbb{R}$, there exists

- *a Carnot group G with a projection $\pi\colon G \to H$*
- $\lambda \in \mathfrak{g}^*$
- $k \in \mathbb{N}$

such that $Q \circ \pi\colon G \to \mathbb{R}$ is a factor of the polynomial $x \mapsto \lambda(\mathrm{Ad}_x Y)$ for every $Y \in \mathfrak{g}^{[k]}$.

Abnormal factors proof

Consider a linear system

$$P_i^{\lambda} = Q \cdot S_i^{\nu}, \quad i = 1, \ldots, m$$

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Abnormal factors proof

Consider a linear system

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \dots, m$$

in the variables (λ, ν), where

- $P_i^\lambda(x) = \lambda(\mathrm{Ad}_x Y_i)$ for a basis $Y_1, \dots, Y_m$ of $\mathfrak{g}^{[k]}$
- S_i^ν are generic polynomials of the form

$$S^\nu = \nu_0 + \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 + \nu_4 x_1^2 + \nu_5 x_1 x_2 + \nu_6 x_2^2 + \dots$$

such that $\deg(S_i^\nu) + \deg(Q) = \deg(P_i)$.

Abnormal factors proof

Let

- $k = \deg Q + 1$
- G_s a free Carnot group of step s

Lemma

The linear system

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \ldots, m$$

has a non-trivial solution (λ, ν) *in* G_s *for large* s*.*

Monomial counting

Proof of Lemma:

1. Hall basis argument $\implies \exists \lambda = \lambda(\nu)$ such that $P_1^{\lambda(\nu)} = Q \cdot S_1^{\nu}$
   Consider the remaining system

$$P_i^{\lambda(\nu)} = Q \cdot S_i^{\nu}, \quad i = 2, \dots, m$$

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   Consider the remaining system
   $$P_i^{\lambda(\nu)} = Q \cdot S_i^{\nu}, \quad i = 2, \ldots, m$$
2. In step s, $\deg(P_i^{\lambda}) \leq s - k$. The number of equations is
   $$(m-1) \cdot \#\{\text{monomials of degree up to } s-k\}$$
   and the number of variables is
   $$m \cdot \#\{\text{monomials of degree up to } s-k-\deg(Q)\}$$

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3. Poincaré series asymptotics for $s \to \infty$
   $\implies$ #variables $\gg$ #equations.

The entire proof

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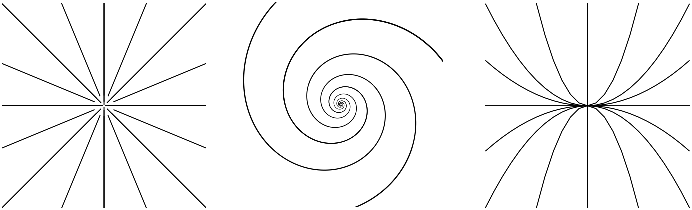
Proof:

1. Every polynomial ODE has a polynomial first integral in a lift.
   - Consider an orthogonal vector field.
   - Every polynomial vector field is a horizontal gradient.
2. Curves contained in an algebraic variety are abnormal in a lift.
   - Common factors of abnormal polynomials = linear system.
   - Monomial counting $\implies$ the system is underdetermined.

Abnormals from linear ODEs

Abnormals in the free Carnot group of rank 2 and step 7



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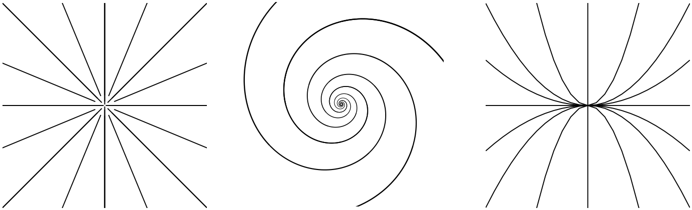

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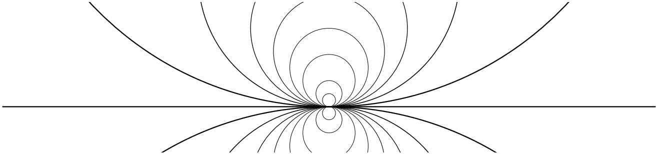
$\exists \lambda \colon \mathbb{R}^6 \to \mathfrak{g}^*$ semi-algebraic such that trajectories of

$$\dot{x} = ax + by + c \quad \dot{y} = dx + ey + f$$

are abnormal with covector $\lambda(a, b, c, d, e, f)$.

Abnormals from quadratic ODEs

Abnormals in the free Carnot group of rank 2 and step 13



$$\dot{z} = z^2, \quad z \in \mathbb{C}$$

Abnormals from quadratic ODEs

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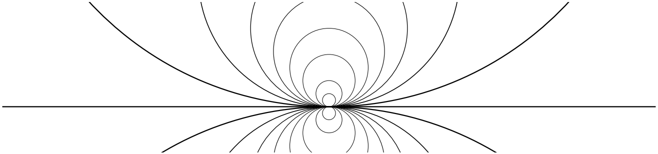

$$\dot{z} = z^2, \quad z \in \mathbb{C}$$

Let $E \subset [0,1]$ be nowhere dense. $\exists$ abnormal curve that is

- injective
- parametrized by arc length on $[0,1] \setminus E$
- not C^2 at any point $x \in E$
- if E is perfect, not C^1 at any point $x \in E$

Thank you for your attention!