Sub-Riemannian manifolds and their abnormal curves

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Sub-Riemannian manifolds

A sub-Riemannian manifold consists of

- a smooth manifold M
- a distribution $\Delta \subset TM$, which is bracket-generating:

$$\Delta + [\Delta, \Delta] + [\Delta, [\Delta, \Delta]] + \cdots = TM$$

- a smoothly varying inner product $\langle \cdot, \cdot \rangle$ on Δ

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Important quantities:

- rank = rank of $\Delta = \dim(\Delta \cap T_pM)$, $p \in M$
- step = minimal length of Lie brackets needed to span TM

Example: standard contact structure on $\mathbb{R}^3$

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equip with a sub-Riemannian metric

- $\langle \cdot, \cdot \rangle$ defined by declaring X and Y orthonormal

Sub-Riemannian path-distance

An absolutely continuous curve $\gamma\colon (a,b) \to M$ such that $\dot\gamma \in \Delta$ is called *horizontal*.

$$d_{SR}(x,y) = \inf\left\{ \int_0^1 \|\dot\gamma\| : \gamma \text{ horizontal}, \gamma(0) = x, \gamma(1) = y \right\}$$

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Theorem (Chow 1939; Rashevsky 1938)

Δ *bracket-generating* $\implies d_{SR} < \infty$.

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Fact

Lipschitz curves $\gamma\colon (a,b) \to (M, d_{SR})$ *are horizontal.*

Parametrizing the horizontal curves

Assume $\Delta = \mathrm{span}\{X_1, \ldots, X_r\}$ and $X_1, \ldots, X_r$ orthonormal.

Fix a base point $p \in M$. A horizontal curve $\gamma\colon [0,1] \to M$ starting from $\gamma(0) = p$ is uniquely characterized by

$$u_i(t) = \langle \dot\gamma(t), X_i \rangle\,, \quad i = 1, \ldots, r.$$

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Definition (Control)

$u = (u_1, \dots, u_r)\colon [0,1] \to \mathbb{R}^r$ is called the *control* of γ.

Note: $\|u\| = \|\dot\gamma\|$.

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Definition (Endpoint map)

The *endpoint map* is the map

$$\text{End} \colon L^2([0,1]; \mathbb{R}^r) \to M, \quad u \mapsto \gamma_u(1).$$

Abnormal curves

Abnormal $\leftrightarrow$ critical points and values of the endpoint map.

Abnormal control $=$ critical point $u \in L^2$ of the endpoint map
Abnormal curve $=$ integral curve γ_u of an abnormal control u
Abnormal set $=$ the set of critical values of the endpoint map

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Abnormal set $=$ the set of critical values of the endpoint map
$=$ the subset of M that can be reached from the basepoint with an abnormal curve.

Open problems

Conjecture (Sard)

The abnormal set has zero measure.

Conjecture (Regularity)

All length-minimizing curves are smooth.

Length-minimality and abnormality

Fact

Suppose $\gamma\colon [0,1] \to M$ is length-minimizing and $\|\dot\gamma\|$ is constant. Then its control u is a critical point of

$$\widetilde{\mathrm{End}}\colon L^2([0,1];\mathbb{R}^r) \to M \times \mathbb{R}, \quad \widetilde{\mathrm{End}}(v) = \left(\mathrm{End}(v), \int_0^1 \|v\|^2\right)$$

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If u is not abnormal, then u satisfies

$$\langle u, v\rangle_{L^2} + \lambda(d_u \,\mathrm{End}(v)) = 0, \quad \lambda \in T^*M.$$

Needle variations $v \implies u$ satisfies a smooth ODE $\implies u$ is C^∞.

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Needle variations $v \implies u$ satisfies a smooth ODE $\implies u$ is C^∞.

Conjecture (Regularity)

All length-minimizing abnormal curves are smooth.

Abnormal dynamics

Theorem (Liu and Sussman 1995)

In sub-Riemannian manifolds of rank 2 and step 3, abnormal length-minimizers have C^∞ regularity.

Theorem (Barilari, Chitour, Jean, Prandi, and Sigalotti 2020)

In sub-Riemannian manifolds of rank 2 and step 4, abnormal length-minimizers have C^1 regularity.

Theorem (Boarotto and Vittone 2020)

In Carnot groups of rank 3 step 3, or rank 2 step 4, the abnormal set is a sub-analytic set with codimension≥ 1.

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Strategy of the proofs:

1. Abnormality can be restated as a dynamical system.
2. Study the dynamics explicitly by reducing to various normal forms.

Abnormal dynamics and complexity

Theorem (H. 2020)

For every polynomial ODE system in $\mathbb{R}^r$, there exists a rank r sub-Riemannian structure on $\mathbb{R}^n$ such that all trajectories of the ODE lift to abnormal curves.

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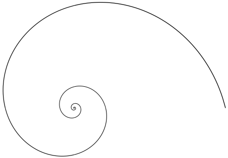
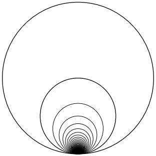
The lift of a curve $x = (x_1, \ldots, x_r)\colon [0,1] \to \mathbb{R}^r$ is the horizontal curve γ_u with control $u = (\dot{x}_1, \ldots, \dot{x}_r)$.

Abnormal dynamics and complexity

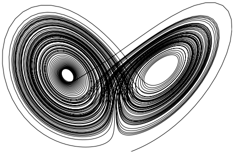
Abnormal curves can be as complicated as trajectories of polynomial ODE systems.



$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = x_1 - x_2$$



$$\dot{z} = z^2, \quad z \in \mathbb{C}$$



$$\dot{x}_1 = 10(x_2 - x_1)$$
$$\dot{x}_2 = 28x_1 - x_2 - x_1x_3$$
$$\dot{x}_3 = x_1x_2 - \frac{8}{3}x_3$$

Types of regularity results for length-minimizing curves

- Results that avoid the presence of abnormals.
- Results that do not use abnormality in any way.
- Results that specifically study abnormals.

Types of regularity results for length-minimizing curves

Results that avoid the presence of abnormals:

- Strichartz 1986: C^∞-regularity for strongly bracket generating structures ($\Delta + [X, \Delta] = TM$ for any horizontal X)
- Chitour, Jean, and Trélat 2006: C^∞-regularity for a generic family of distributions Δ with $\dim \Delta_p \geq 3$.

Types of regularity results for length-minimizing curves

Results that do not use abnormality in any way:

- H. and Le Donne 2016: no corner-type singularities
- Monti, Pigati, and Vittone 2018: existence of tangent lines
- Monti and Socionovo 2021: no spiral-type singularities for sub-Riemannian structures where $[[\Delta, \Delta], [\Delta, \Delta]] = 0$.
- H. and Le Donne 2022: iterated metric blowups or blowdowns are lines

Types of regularity results for length-minimizing curves

Results that specifically study abnormals:

- Liu and Sussman 1995: C^∞-regularity for a class of abnormal curves, called the regular abnormal extremals
- Sussmann 2014: analytic regularity on an open dense subset, if the sub-Riemannian structure is analytic
- Belotto da Silva, Figalli, Parusiński, and Rifford 2018: C^1-regularity for 3-dimensional analytic sub-Riemannian manifolds
- Barilari, Chitour, Jean, Prandi, and Sigalotti 2020: C^1-regularity for a class of abnormal curves generalizing the regular abnormal extremals

Some Sard results

Assume the sub-Riemannian structure is analytic.

Then the abnormal set is ...

- ...contained in a closed nowhere dense set (Agrachëv 2009)
- ...a countable union of semianalytic curves in 3d SR manifolds (Belotto da Silva, Figalli, Parusiński, and Rifford 2018)
- ...a proper algebraic subvariety in Carnot groups of step 2, in $\mathbb{F}_{2,4}$, and in $\mathbb{F}_{3,3}$ (Le Donne, Montgomery, Ottazzi, Pansu, and Vittone 2016)
- ...a proper sub-analytic subvariety in Carnot groups of rank 3 step 3, and in rank 2 step 4 (Boarotto and Vittone 2020)

Carnot groups

A *Carnot group* is a Lie group G whose Lie algebra $\mathfrak{g}$ is stratified:

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}$$

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A sub-Riemannian structure on a Carnot group G:

- Δ left-invariant with $\Delta_e = \mathfrak{g}^{[1]}$.
- $\langle \cdot, \cdot \rangle$ left-invariant.

The basepoint p is the identity element $e \in G$.

Infinitesimal structure of SR manifolds

Theorem (Mitchell 1985)

The metric tangent of an equiregular sub-Riemannian manifold is a sub-Riemannian Carnot group.

$$\text{equiregular} \iff \begin{array}{l} p \mapsto \dim[\Delta, \Delta]_p \\ p \mapsto \dim[\Delta, [\Delta, \Delta]]_p \\ p \mapsto \dim[\Delta, [\Delta, [\ldots, \Delta] \ldots]]_p \end{array} \quad \text{are all constant.}$$

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Non-equiregular distribution $\Delta = \text{span}\{X, Y\} \subset T\mathbb{R}^3$:

$$\begin{aligned} X &= \partial_x \\ Y &= \partial_y + x^2 \partial_z \\ [X, Y] &= 2x\partial_z \\ [X, [X, Y]] &= 2\partial_z \end{aligned}$$

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Theorem (Bellaïche 1996)

The metric tangent of any sub-Riemannian manifold is a sub-Riemannian homogeneous space, that is, a quotient G/H of sub-Riemannian Carnot groups.

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$$n\text{-manifolds} \leftrightsquigarrow \mathbb{R}^n$$

$$\text{sub-Riemannian manifolds} \leftrightsquigarrow \text{sub-Riemannian Carnot groups}$$

Free Carnot groups

Consider the free Lie algebra $\mathfrak{f}_r$ with generators $X_1, \ldots, X_r$, so the only relations are generated by

$$[X, Y] + [Y, X] = 0 \quad \text{and}$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

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Denote $\mathfrak{f}_r^1 = \operatorname{span}\{X_1, \ldots, X_r\}$ and $\mathfrak{f}_r^{[k+1]} = [\mathfrak{f}_r^{[1]}, \mathfrak{f}_r^{[k]}]$.

Definition (Free Carnot group)

The free Carnot group $\mathbb{F}_{r,s}$ of rank r and step s is the Carnot group with Lie algebra $\mathfrak{f}_r/(\mathfrak{f}_r^{[s+1]} \oplus \mathfrak{f}_r^{[s+2]} \oplus \ldots)$.

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$$[X_1, [X_1, \dots, [X_1, X_2]]] \neq 0$$
$$[X_1, [X_1, [X_1, \dots, [X_1, X_2]]]] = 0$$

A tower of increasing complexity

The free Carnot groups have projections

$$\mathbb{R}^r = \mathbb{F}_{r,1} \leftarrow \mathbb{F}_{r,2} \leftarrow \mathbb{F}_{r,3} \leftarrow \cdots$$

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Horizontal curves in $\mathbb{F}_{r,s}$ lift to horizontal curves in $\mathbb{F}_{r,\tilde{s}}$, $\tilde{s} > s$.

Fact

Abnormality is preserved by the lift.

Characterization of abnormals

G Carnot group with Lie algebra $\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}$.

Lemma

$\gamma\colon [0,1] \to G$ *abnormal* $\iff \lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[1]}) = 0$ *for some* $\lambda \in \mathfrak{g}^*$.

$$\mathrm{Ad}_\gamma \colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_\gamma X = \frac{d}{ds}\, \gamma \cdot \exp(sX) \cdot \gamma^{-1}\Big|_{s=0}$$

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Fact

In coordinates $(x_1, \ldots, x_n)$ *on* G, *the map*

$$\mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto \lambda(\mathrm{Ad}_{(x_1,\ldots,x_n)} X)$$

is a polynomial for every $X \in \mathfrak{g}$ *and* $\lambda \colon \mathfrak{g} \to \mathbb{R}$.

Horizontal gradients

Theorem (H. 2020)

For every polynomial ODE system in $\mathbb{R}^r$, there exists a rank r sub-Riemannian structure on $\mathbb{R}^n$ such that all trajectories of the ODE lift to abnormal curves.

Definition (Lift)

The lift of a curve $x = (x_1, \dots, x_r)\colon [0,1] \to \mathbb{R}^r$ is the horizontal curve γ_u with control $u = (\dot{x}_1, \dots, \dot{x}_r)$.

Proof idea:

1. Every polynomial ODE is a horizontal gradient...
2. Curves contained in an algebraic variety are abnormal...

... in some lift.

Gradients in $\mathbb{R}^r$

$$P = (P_1, \ldots, P_r) = \nabla Q \text{ for some } Q\colon \mathbb{R}^r \to \mathbb{R} \iff \partial_i P_j = \partial_j P_i$$

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Recursion for Q:

$$\begin{aligned} Q_1 &= \int P_1 \, dx_1 \\ Q_2 &= Q_1 + \int (P_2 - \partial_2 Q_1) \, dx_2 \\ &\vdots \\ Q = Q_r &= Q_{r-1} + \int (P_r - \partial_r Q_{r-1}) \, dx_r \end{aligned}$$

A non-gradient vector field in $\mathbb{R}^2$

$$P(x) = (x_1 - x_2, x_1 + x_2) \neq (\partial_1 Q(x), \partial_2 Q(x)) = \nabla Q(x)$$

for any polynomial $Q\colon \mathbb{R}^2 \to \mathbb{R}$, since

$$\partial_1 P_2 = 1 \neq -1 = \partial_2 P_1, \quad \text{but} \quad \partial_1 \partial_2 Q = \partial_2 \partial_1 Q.$$

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Lift P to a horizontal vector field in the Heisenberg group.

$$\begin{aligned} X_1(x) &= \partial_1 \\ X_2(x) &= \partial_2 + x_1 \partial_3 \\ X_3(x) &= [X_1, X_2](x) = \partial_3 \end{aligned}$$

$$P\colon H \to TH, \quad P(x) = (x_1 - x_2) X_1(x) + (x_1 + x_2) X_2(x)$$

Recursion for horizontal gradient integration

Suppose $P = \nabla_{\text{hor}} Q = (X_1 Q) X_1 + (X_2 Q) X_2$. Then

$$X_1 Q = x_1 - x_2$$
$$X_2 Q = x_1 + x_2$$
$$X_3 Q = [X_1, X_2] Q = X_1(X_2 Q) - X_2(X_1 Q) = 2$$

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Integrate backwards:

$$\begin{aligned} Q_3 &= \int X_3 Q \, dx_3 \\ Q_2 &= Q_3 + \int (X_2 Q - X_2 Q_3) \, dx_2 \\ Q = Q_1 &= Q_2 + \int (X_1 Q - X_1 Q_2) \, dx_1 \\ &= \frac{1}{2}x_1^2 - x_1 x_2 + \frac{1}{2}x_2^2 + 2x_3 \end{aligned}$$

Horizontal gradients

Lemma

Every polynomial vector field $P\colon \mathbb{R}^r \to \mathbb{R}^r$ is the horizontal gradient of some polynomial in a Carnot group of high enough step.

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For the frame $X_1, \ldots, X_r$ the horizontal gradient of $Q\colon G \to \mathbb{R}$ is

$$\nabla_{\text{hor}} Q = \sum (X_i Q) X_i \colon G \to TG.$$

In coordinates, lift $P\colon \mathbb{R}^r \to \mathbb{R}^r$ to the horizontal vector field

$$P\colon G \to TG, \quad P(x_1, \ldots, x_r, \ldots, x_n) = \sum_{i=1}^{r} P_i(x_1, \ldots, x_r) X_i(x)$$

Horizontal gradients

Why it works:

- As *weighted* differential operators, $[X_1, X_2]$ is a degree 2 operator, $[X_1, [X_1, X_2]]$ is degree 3, etc.
  $\implies$ partial derivatives of a polynomial eventually vanish
- There exist coordinates such that $X_i = \partial_i + \sum_{j>i} c_{ij}\partial_j$.
  $\implies$ integration variable by variable is possible

A horizontal first integral

For an ODE

$$\dot{x}_i = P_i(x), \quad x \in \mathbb{R}^r, \quad i = 1, \dots, n$$

integrate any nonzero orthogonal vector field.

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$$X_1 Q = -P_2, \quad X_2 Q = P_1 \quad X_3 Q = X_4 Q = \cdots = X_r Q = 0.$$

Then for a trajectory $x\colon [0,1] \to G$ of $\dot{x} = \sum P_i(x) X_i(x)$

$$\frac{d}{dt} Q(x) = P_1(x) X_1 Q(x) + \cdots + P_r(x) X_r Q(x) = 0.$$

Abnormal factors

Theorem (H. 2020)

For every polynomial ODE system in $\mathbb{R}^r$, there exists a rank r sub-Riemannian structure on $\mathbb{R}^n$ such that all trajectories of the ODE lift to abnormal curves.

Definition (Lift)

The lift of a curve $x = (x_1, \ldots, x_r) \colon [0,1] \to \mathbb{R}^r$ is the horizontal curve γ_u with control $u = (\dot{x}_1, \ldots, \dot{x}_r)$.

Proof idea:

1. Every polynomial ODE is a horizontal gradient...
2. Curves contained in an algebraic variety are abnormal...

... in some lift.

Higher order abnormality

$$\mathfrak{g} = \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \cdots \oplus \mathfrak{g}^{[s]}, \quad [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}] = \mathfrak{g}^{[i+1]}.$$

Definition

$\gamma\colon [0,1] \to G$ abnormal $\iff \lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[1]}) = 0$

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Lemma

If $\gamma(0) = e$ and $\lambda(\mathrm{Ad}_{\gamma(t)}\, \mathfrak{g}^{[k]}) = 0$, then γ is abnormal of order k.

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Proposition

For any polynomial $Q\colon H \to \mathbb{R}$*, there exists*

- *a Carnot group* G *with a projection* $\pi\colon G \to H$
- $\lambda \in \mathfrak{g}^*$
- $k \in \mathbb{N}$

such that $Q \circ \pi\colon G \to \mathbb{R}$ *is a factor of the polynomial* $x \mapsto \lambda(\mathrm{Ad}_x Y)$ *for every* $Y \in \mathfrak{g}^{[k]}$.

Abnormal factors proof

Consider a linear system

$$P_i^\lambda = Q \cdot S_i^\nu, \quad i = 1, \ldots, m \tag{1}$$

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in the variables (λ, ν), where

- $P_i^\lambda(x) = \lambda(\mathrm{Ad}_x Y_i)$ for a basis $Y_1, \dots, Y_m$ of $\mathfrak{g}^{[k]}$
- $k = \deg Q + 1$
- S_i^ν are generic polynomials of the form

$$S^\nu = \nu_0 + \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 + \nu_4 x_1^2 + \nu_5 x_1 x_2 + \nu_6 x_2^2 + \dots$$

such that $\deg(S_i^\nu) + \deg(Q) = \deg(P_i)$.

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  such that $\deg(S_i^\nu) + \deg(Q) = \deg(P_i)$.

Lemma

The linear system (1) has a non-trivial solution (λ, ν) in $\mathbb{F}_{r,s}$ for large enough s.

Monomial counting

Proof of Lemma:

1. Hall basis argument $\implies \exists\lambda = \lambda(\nu)$ such that $P_1^{\lambda(\nu)} = Q \cdot S_1^{\nu}$
Consider the remaining system

$$P_i^{\lambda(\nu)} = Q \cdot S_i^{\nu}, \quad i = 2, \dots, m$$

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   $$P_i^{\lambda(\nu)} = Q \cdot S_i^{\nu}, \quad i = 2, \dots, m$$
2. In step s, $\deg(P_i^{\lambda}) \leq s - k$. The number of equations is
   $$(m-1) \cdot \#\{\text{monomials of degree up to } s-k\}$$
   and the number of variables is
   $$m \cdot \#\{\text{monomials of degree up to } s-k-\deg(Q)\}$$

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3. Poincaré series asymptotics for $s \to \infty$
   $\implies \#\text{variables} \gg \#\text{equations}$.

The entire proof

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For every polynomial ODE system in $\mathbb{R}^r$, there exists a rank r sub-Riemannian structure on $\mathbb{R}^n$ such that all trajectories of the ODE lift to abnormal curves.

Proof:

1. Every polynomial ODE has a polynomial first integral in a lift.
   - Consider an orthogonal vector field.
   - Every polynomial vector field is a horizontal gradient.
2. Curves contained in an algebraic variety are abnormal in a lift.
   - Common factors of abnormal polynomials $=$ linear system.
   - Monomial counting $\implies$ the system is underdetermined.

An inefficient formula

Let $P \colon \mathbb{R}^r \to \mathbb{R}^r$ be a polynomial vector field.

Let $d(r,k) = \dim \mathfrak{f}_r^{[k]} = \dfrac{1}{k}\sum_{d|k} \mu(d) r^{k/d}$.

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Consider the rational function

$$\sum_{k=0}^{\infty} C_k t^k = \frac{\Big(1-(d(r,\deg(P)+1))(1-t^{\deg(P)})\Big)\, t^{\deg(P)+1}}{\prod_{k=1}^{\deg(P)}(1-t^k)^{d(r,k)}}.$$

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If $\sum_{k=0}^{s} C_k > 0$, then trajectories of P are abnormal in step s.

Inefficient numbers from an inefficient formula

P a polynomial vector field in $\mathbb{R}^r$. Trajectories abnormal in step:

$r\backslash\deg(P)$	1	2	3	4	5
2	11	38	172	577	2372
3	89	724	6034	46036	365813
4	386	5322	73109	983505	13529000

Example

A polynomial vector field in $\mathbb{R}^4$ of degree 5 has abnormal lifts in the free Carnot group G of rank 4 and step 13529000.

$$\dim G \approx 4.1338 \cdot 10^{8145262}$$

Experimental abnormality steps

Theoretical bounds:

$r \backslash \deg(P)$	1	2	3	4	5
2	11	38	172	577	2372
3	89	724	6034	46036	365813
4	386	5322	73109	983505	13529000

Bounds in randomly sampled homogeneous ODE examples:

$r \backslash \deg(P)$	1	2	3	4	5
2	7	13	19		
3	7	13			
4	7				

Thank you for your attention!