Isometric embeddings in Carnot groups of step 2

Eero Hakavuori

University of Jyväskylä

May 13, 2019

Isometric embeddings

Theorem

If $(V, \|\cdot\|)$ *is a normed space with*

- $\|\cdot\|$ *strictly convex,*

then

- *geodesics* $[a, b] \hookrightarrow V$ *are line segments*
- *isometric embeddings* $W \hookrightarrow V$ *are affine.*

Isometric embeddings

Theorem

If $(G, \|\cdot\|)$ is a sub-Finsler Carnot group with

- *G step 2*
- *$\|\cdot\|$ strictly convex,*

then

- *infinite geodesics $\mathbb{R} \hookrightarrow G$ are lines*
- *isometric embeddings $H \hookrightarrow G$ are affine.*

Isometric embeddings

Theorem

If $(G, \|\cdot\|)$ *is a sub-Finsler Carnot group with*

- G *step 2*
- $\|\cdot\|$ *strictly convex,*

then

- *infinite geodesics* $\mathbb{R} \hookrightarrow G$ *are lines*
- *isometric embeddings* $H \hookrightarrow G$ *are affine.*

Carnot groups

Definition

A *sub-Finsler Carnot group* consists of

- a Lie group G,
- a *horizontal subspace* $V_1 \subset \mathfrak{g}$, and
- a norm $\|\cdot\|: V_1 \to \mathbb{R}$

such that $\mathfrak{g}$ is stratified:

$$V_{k+1} := [V_1, V_k] \implies \begin{cases} \mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s \\ V_{s+1} = \{0\} \end{cases} .$$

Carnot groups

Definition

A *sub-Finsler Carnot group* consists of

- a Lie group G,
- a *horizontal subspace* $V_1 \subset \mathfrak{g}$, and
- a norm $\|\cdot\|\colon V_1 \to \mathbb{R}$

such that $\mathfrak{g}$ is stratified:

$$V_{k+1} := [V_1, V_k] \implies \begin{cases} \mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s \\ V_{s+1} = \{0\} \end{cases} .$$

step 2 $\iff$ $\mathfrak{g} = V_1 \oplus V_2$

Metric and dilations

- A left-invariant geodesic metric:

$$d(g,h) = \min \left\{ \int_0^1 \|\dot\gamma(t)\|\, dt : \dot\gamma(t) \in V_1,\ \begin{array}{c} \gamma(0) = g \\ \gamma(1) = h \end{array} \right\}$$

- $\mathfrak{g}$ stratified $\implies \exists$ dilations $\delta_\lambda \colon G \to G$:

$$\delta(\exp(\underset{V_1}{\underset{\cap}{X_1}} + \cdots + \underset{V_s}{\underset{\cap}{X_s}})) = \exp(\lambda X_1 + \cdots + \lambda^s X_s)$$

$$d(\delta_\lambda(g), \delta_\lambda(h)) = \lambda d(g,h).$$

Affinity

In a normed space V:

Definition

affine $=$ composition of a translation and a linear map

Definition

$$\begin{aligned} \textit{line} &= t \mapsto y + tx \\ &= \text{a curve through } y \in V \text{ with constant derivative } x \in V \end{aligned}$$

Affinity

In a Carnot group G:

Definition

affine = composition of a left translation and a homomorphism

Definition

$$\begin{aligned} \textit{line} &= t \mapsto g\exp(tX) \\ &= \text{a curve through } g \in G \text{ with constant derivative } X \in \mathfrak{g} \end{aligned}$$

The horizontal space

- V_1 generates $\mathfrak{g} \implies \exp(V_1)$ generates G:

$$g \in G \implies \exists g_1, \ldots, g_m \in \exp(V_1): \; g = g_1 g_2 \ldots g_m.$$

Definition (Horizontal projection)

$$\pi : G \to V_1, \qquad \exp(\underset{V_1}{\underset{\Cap}{X_1}} + \underset{V_2}{\underset{\Cap}{X_2}} + \cdots + \underset{V_s}{\underset{\Cap}{X_s}}) \mapsto X_1$$

- The horizontal projection is a submetry

$$\pi(B_d(g, r)) = B_{\|\cdot\|}(\pi(g), r)$$

and a homomorphism

$$\pi(gh) = \pi(g) + \pi(h).$$

Isometric embeddings

Theorem

If $(G, \|\cdot\|)$ is a sub-Finsler Carnot group with

- *G step 2*
- *$\|\cdot\|$ strictly convex,*

then

- *infinite geodesics $\mathbb{R} \hookrightarrow G$ are lines*
- *isometric embeddings $H \hookrightarrow G$ are affine.*

Isometric embeddings

Theorem

If $(G, \|\cdot\|)$ is a sub-Finsler Carnot group with

- *G step 2*
- *$\|\cdot\|$ strictly convex,*

then

- *infinite geodesics $\mathbb{R} \hookrightarrow G$ are lines*
- *isometric embeddings $H \hookrightarrow G$ are affine.*

In fact, in step 2:

$\|\cdot\|$ strictly convex $\iff$ infinite geodesics are lines
$\iff$ isometric embeddings are affine

The immediate implications

$\|\cdot\|$ strictly convex $\Longleftarrow$ infinite geodesics are lines

$\Longleftarrow$ isometric embeddings are affine

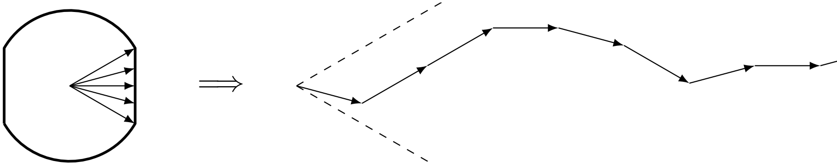
The immediate implications

$\|\cdot\|$ not strictly convex $\Longrightarrow$ $\exists$ non-line infinite geodesics

- In a normed space:

unit ball

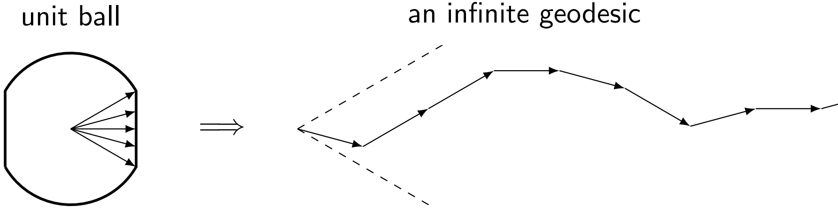
an infinite geodesic



The immediate implications

$\|\cdot\|$ not strictly convex $\implies$ $\exists$ non-line infinite geodesics

- In a normed space:



- In a Carnot group:

geodesic in $(V_1, \|\cdot\|)$ $\overset{\text{lift}}{\implies}$ geodesic in G

From geodesics to embeddings

Theorem

In every sub-Finsler Carnot group G:

$$\begin{matrix} \textit{infinite geodesics} \\ \mathbb{R} \hookrightarrow G \textit{ are lines} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \textit{isometric embeddings} \\ H \hookrightarrow G \textit{ are affine.} \end{matrix}$$

From geodesics to embeddings

Theorem

In every sub-Finsler Carnot group G:

$$\begin{matrix} \textit{infinite geodesics} & & \textit{isometric embeddings} \\ \mathbb{R} \hookrightarrow G \textit{ are lines} & \Longrightarrow & H \hookrightarrow G \textit{ are affine.} \end{matrix}$$

- Balogh, Fässler and Sobrino (2018) for Heisenberg groups.

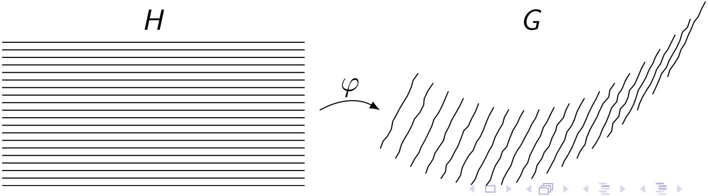
From geodesics to embeddings

Theorem

In every sub-Finsler Carnot group G:

$$\begin{matrix}\textit{infinite geodesics} \\ \mathbb{R} \hookrightarrow G \textit{ are lines}\end{matrix} \implies \begin{matrix}\textit{isometric embeddings} \\ H \hookrightarrow G \textit{ are affine.}\end{matrix}$$

- Balogh, Fässler and Sobrino (2018) for Heisenberg groups.
- Idea:
  1. $\varphi\colon H \hookrightarrow G$ isometric embedding, $\varphi(1_H) = 1_G$.
  2. Foliate H by horizontal lines $t \mapsto h \exp_H(tX)$, $h \in H$, $X \in V_1^H$.
  3. Study foliation of $\varphi(H)$ by infinite geodesics $t \mapsto \varphi(h \exp_H(tX))$.



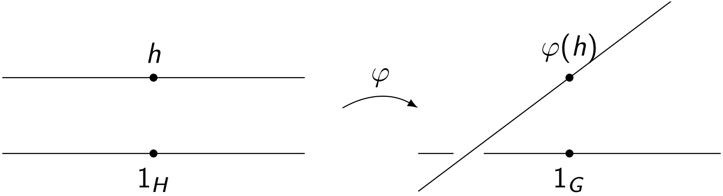
From geodesics to embeddings

3. Study foliation by infinite geodesics $t \mapsto \varphi(h \exp_H(tX))$.

From geodesics to embeddings

3. Study foliation by infinite geodesics $t \mapsto \varphi(h \exp_H(tX))$.
   - Geodesic linearity $\implies$ φ maps lines to lines:

$$\varphi(\exp_H(tX)) = \exp_G(tY), \quad Y \in V_1^G$$
$$\varphi(h \exp_H(tX)) = \varphi(h) \exp_G(tZ), \quad Z \in V_1^G.$$



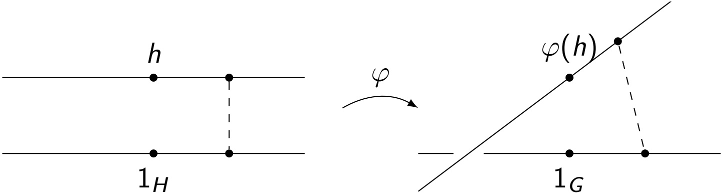
From geodesics to embeddings

3. Study foliation by infinite geodesics $t \mapsto \varphi(h \exp_H(tX))$.
   - Geodesic linearity $\implies$ φ maps lines to lines:

$$\varphi(\exp_H(tX)) = \exp_G(tY), \quad Y \in V_1^G$$
$$\varphi(h \exp_H(tX)) = \varphi(h) \exp_G(tZ), \quad Z \in V_1^G.$$

   - Consider the distance

$$d(\exp_H(tX), h \exp_H(tX)) = d(\exp_G(tY), \varphi(h) \exp_G(tZ)).$$



From geodesics to embeddings

- Parallel lines diverge sublinearly:

Lemma

In a Carnot group $\forall h \in G \ \forall A, B \in V_1$

$$d(\exp(tA), h\exp(tB)) = o(t) \text{ as } t \to \infty \iff A = B.$$

From geodesics to embeddings

- Parallel lines diverge sublinearly:

Lemma

In a Carnot group $\forall h \in G\ \forall A, B \in V_1$

$$d(\exp(tA), h\exp(tB)) = o(t) \text{ as } t \to \infty \iff A = B.$$

- $d(\exp_H(tX), h\exp_H(tX)) = d(\exp_G(tY), \varphi(h)\exp_G(tZ)) \implies Y = Z$

From geodesics to embeddings

- Parallel lines diverge sublinearly:

Lemma

In a Carnot group $\forall h \in G\ \forall A, B \in V_1$

$$d(\exp(tA), h\exp(tB)) = o(t) \text{ as } t \to \infty \iff A = B.$$

- $d(\exp_H(tX), h\exp_H(tX)) = d(\exp_G(tY), \varphi(h)\exp_G(tZ))$
  $\implies Y = Z$
  $\implies \varphi(h\exp_H(tX)) = \varphi(h)\varphi(\exp_H(tX)).$

From geodesics to embeddings

- Parallel lines diverge sublinearly:

Lemma

In a Carnot group $\forall h \in G\ \forall A, B \in V_1$

$$d(\exp(tA), h\exp(tB)) = o(t) \text{ as } t \to \infty \iff A = B.$$

- $d(\exp_H(tX), h\exp_H(tX)) = d(\exp_G(tY), \varphi(h)\exp_G(tZ))$
  $\implies Y = Z$
  $\implies \varphi(h\exp_H(tX)) = \varphi(h)\varphi(\exp_H(tX)).$
- $\varphi(hg) = \varphi(h)\varphi(g)\ \forall h \in H\ \forall g \in \exp(V_1^H).$

From geodesics to embeddings

- Parallel lines diverge sublinearly:

Lemma

In a Carnot group $\forall h \in G\ \forall A, B \in V_1$

$$d(\exp(tA), h\exp(tB)) = o(t) \text{ as } t \to \infty \iff A = B.$$

- $d(\exp_H(tX), h\exp_H(tX)) = d(\exp_G(tY), \varphi(h)\exp_G(tZ))$
  $\implies Y = Z$
  $\implies \varphi(h\exp_H(tX)) = \varphi(h)\varphi(\exp_H(tX))$.
- $\varphi(hg) = \varphi(h)\varphi(g)\ \forall h \in H\ \forall g \in \exp(V_1^H)$.
- $\exp(V_1^H)$ generates $H \implies \varphi$ is a homomorphism.

Infinite geodesics

Theorem

$(G, \|\cdot\|)$ sub-Finsler Carnot group of step 2.

$\|\cdot\|$ strictly convex $\implies$ infinite geodesics $\gamma \colon \mathbb{R} \hookrightarrow G$ are lines.

Infinite geodesics

Theorem

$(G, \|\cdot\|)$ sub-Finsler Carnot group of step 2.
$\|\cdot\|$ strictly convex $\implies$ infinite geodesics $\gamma\colon \mathbb{R} \hookrightarrow G$ are lines.

Proof structure:

1. Pontryagin Maximum Principle $\rightsquigarrow$ implicit geodesic equation.

Infinite geodesics

Theorem

$(G, \|\cdot\|)$ sub-Finsler Carnot group of step 2.
$\|\cdot\|$ strictly convex $\implies$ infinite geodesics $\gamma : \mathbb{R} \hookrightarrow G$ are lines.

Proof structure:

1. Pontryagin Maximum Principle $\rightsquigarrow$ implicit geodesic equation.
2. The asymptotic cone of γ contains a line $t \mapsto \exp(tY)$.

Infinite geodesics

Theorem

$(G, \|\cdot\|)$ sub-Finsler Carnot group of step 2.
$\|\cdot\|$ strictly convex $\implies$ infinite geodesics $\gamma\colon \mathbb{R} \hookrightarrow G$ are lines.

Proof structure:

1. Pontryagin Maximum Principle $\rightsquigarrow$ implicit geodesic equation.
2. The asymptotic cone of γ contains a line $t \mapsto \exp(tY)$.
3. Implicit geodesic equation $\implies$ both Y and $\dot{\gamma}(t)$ maximize a linear function $a(t)$ in the ball $\{\|\cdot\| \leq 1\}$.

Infinite geodesics

Theorem

$(G, \|\cdot\|)$ *sub-Finsler Carnot group of step 2.*

$\|\cdot\|$ *strictly convex* $\implies$ *infinite geodesics* $\gamma\colon \mathbb{R} \hookrightarrow G$ *are lines.*

Proof structure:

1. Pontryagin Maximum Principle $\rightsquigarrow$ implicit geodesic equation.
2. The asymptotic cone of γ contains a line $t \mapsto \exp(tY)$.
3. Implicit geodesic equation $\implies$ both Y and $\dot\gamma(t)$ maximize a linear function $a(t)$ in the ball $\{\|\cdot\| \leq 1\}$.
4. $$\|\cdot\| \text{ strictly convex } \implies a(t) \text{ has a unique maximum} \implies \dot\gamma(t) \equiv Y$$

Implicit geodesic equation and linear invariants

Proposition (Step 2 sub-Finsler PMP)

For every geodesic $\gamma\colon \mathbb{R} \to G$ in a step 2 sub-Finsler Carnot group, there exist

- *a dual curve $a\colon \mathbb{R} \to V_1^*$, and*
- *a skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$*

such that for a.e. $t \in \mathbb{R}$:

- $\frac{d}{dt}a(t)Y = B(\dot\gamma(t), Y) \quad \forall Y \in V_1$
- *$a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$.*

Implicit geodesic equation and linear invariants

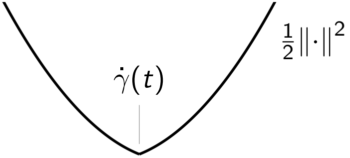
Proposition (Step 2 sub-Finsler PMP)

For every geodesic $\gamma\colon \mathbb{R} \to G$ in a step 2 sub-Finsler Carnot group, there exist

- *a dual curve $a\colon \mathbb{R} \to V_1^*$, and*
- *a skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$*

such that for a.e. $t \in \mathbb{R}$:

- $\frac{d}{dt} a(t)Y = B(\dot\gamma(t), Y) \quad \forall Y \in V_1$
- *$a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$.*



Implicit geodesic equation and linear invariants

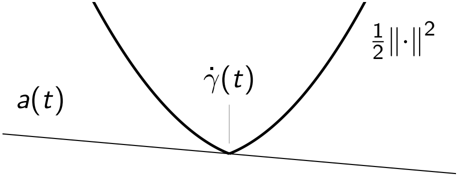
Proposition (Step 2 sub-Finsler PMP)

For every geodesic $\gamma\colon \mathbb{R} \to G$ in a step 2 sub-Finsler Carnot group, there exist

- *a dual curve $a\colon \mathbb{R} \to V_1^*$, and*
- *a skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$*

such that for a.e. $t \in \mathbb{R}$:

- $\frac{d}{dt}a(t)Y = B(\dot\gamma(t), Y) \quad \forall Y \in V_1$
- *$a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$.*



Implicit geodesic equation and linear invariants

Proposition (Step 2 sub-Finsler PMP)

For every geodesic $\gamma\colon \mathbb{R} \to G$ in a step 2 sub-Finsler Carnot group, there exist

- *a dual curve $a\colon \mathbb{R} \to V_1^*$, and*
- *a skew-symmetric bilinear form $B\colon V_1 \times V_1 \to \mathbb{R}$*

such that for a.e. $t \in \mathbb{R}$:

- $\frac{d}{dt}a(t)Y = B(\dot\gamma(t), Y) \quad \forall Y \in V_1$
- *$a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$.*

- $Y \in \ker B \implies \dfrac{d}{dt}a(t)Y = 0$

  $\implies Y\colon V_1^* \to \mathbb{R}$ is invariant along any solution.

The asymptotic invariant

Theorem (H. - Le Donne, 2018)

G sub-Finsler Carnot group with strictly convex norm. $\gamma\colon \mathbb{R} \to G$ *geodesic* $\implies \exists H < G$ *of lower rank such that* $\mathrm{Asymp}(\gamma) \subset H$.

$$\implies \exists \lambda_k \to \infty \text{ such that } \lim_{k\to\infty} \delta_{1/\lambda_k}\gamma(\lambda_k t) = \exp(tY)\ \forall t \in \mathbb{R}.$$

The asymptotic invariant

Theorem (H. - Le Donne, 2018)

G sub-Finsler Carnot group with strictly convex norm. $\gamma : \mathbb{R} \to G$ geodesic $\implies$ $\exists H < G$ of lower rank such that $\mathrm{Asymp}(\gamma) \subset H$.

$$\implies \exists \lambda_k \to \infty \text{ such that } \lim_{k\to\infty} \delta_{1/\lambda_k}\gamma(\lambda_k t) = \exp(tY)\ \forall t \in \mathbb{R}.$$

Remark (Monti - Pigati - Vittone, 2018)

γ_k geodesics, $\gamma_k \to \tilde{\gamma}$ in L^∞_{loc} $\implies$ $\exists$ subsequence $\gamma_k \to \tilde{\gamma}$ in $\mathrm{W}^{1,2}_{loc}$.

$$\implies \lim_{k\to\infty} \dot{\gamma}(\lambda_k t) = Y \text{ for almost every } t \in \mathbb{R}.$$

The asymptotic invariant

Lemma

$$\lambda_k \to \infty \implies Y = \lim_{k\to\infty} \dot{\gamma}(\lambda_k t) \in \ker B$$

The asymptotic invariant

Lemma

$\lambda_k \to \infty \implies Y = \lim_{k\to\infty} \dot{\gamma}(\lambda_k t) \in \ker B$

Proof part 1/3: Suffices to show $\forall t_1 < t_2$

$$\lim_{k\to\infty} \fint_{t_1}^{t_2} \dot{\gamma}(\lambda_k t) \in \ker B.$$

That is:

$$B\left(\fint_{t_1}^{t_2} \dot{\gamma}(\lambda t)\, dt, X\right) \to 0 \quad \text{as } \lambda \to \infty \quad \forall X \in V_1.$$

The asymptotic invariant

Lemma

$$\lambda_k \to \infty \implies Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t) \in \ker B$$

Proof part 2/3:

Proposition (Step 2 sub-Finsler PMP)

...

- $\frac{d}{dt} a(t)X = B(\dot\gamma(t), X) \quad \forall X \in V_1$

$$B\left(\int_{t_1}^{t_2} \dot\gamma(\lambda t)\, dt, X \right) = \int_{\lambda t_1}^{\lambda t_2} B\left(\dot\gamma(t), X\right)\, dt$$
$$= \frac{1}{\lambda(t_2 - t_1)} \left(a(\lambda t_2)X - a(\lambda t_1)X\right)$$

The asymptotic invariant

Lemma

$\lambda_k \to \infty \implies Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t) \in \ker B$

Proof part 3/3:

$$B\left(\int_{t_1}^{t_2} \dot\gamma(\lambda t)\, dt, X\right) = \frac{1}{\lambda(t_2 - t_1)}\left(a(\lambda t_2)X - a(\lambda t_1)X\right)$$

Proposition (Step 2 sub-Finsler PMP)

...

- *$a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$.*

$$\implies |a(t)X| \leq \|\dot\gamma(t)\|\|X\| = \|X\|$$
$$\implies |a(\lambda t_2)X - a(\lambda t_1)X| \leq 2\|X\|.$$

The asymptotic invariant

The asymptotic invariant

$\gamma\colon \mathbb{R} \to G$ geodesic $\implies \exists \lambda_k \to \infty \ \exists Y \in V_1$:

- $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$ for almost every $t \in \mathbb{R}$
- $Y\colon V_1^* \to \mathbb{R}$ is an invariant: $a(t)Y \equiv C$.

Subdifferentials – compactness

The asymptotic invariant

- $Y = \lim_{k \to \infty} \dot{\gamma}(\lambda_k t)$ for almost every $t \in \mathbb{R}$
- $Y : V_1^* \to \mathbb{R}$ is an invariant: $a(t)Y \equiv C$.

Proposition (Step 2 sub-Finsler PMP)

...

- *$a(\lambda_k t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot{\gamma}(\lambda_k t)$.*

Subdifferentials – compactness

The asymptotic invariant

- $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$ for almost every $t \in \mathbb{R}$
- $Y : V_1^* \to \mathbb{R}$ is an invariant: $a(t)Y \equiv C$.

Proposition (Step 2 sub-Finsler PMP)

...

- *$a(\lambda_k t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(\lambda_k t)$.*

$\implies$ $\exists$ subsequence $a(\lambda_k t) \to \tilde a \in V_1^*$ such that

- $\tilde a$ subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$
- $\tilde a Y = C$

Subdifferentials – maximizers

For almost every $t \in \mathbb{R}$:

- $a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$
- $\tilde{a}$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$

Subdifferentials – maximizers

For almost every $t \in \mathbb{R}$:

- $a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$
- $\tilde a$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$

$$\tilde a Y = \max_{\|X\|\le\|Y\|} \tilde a X \qquad \text{and} \qquad a(t)\dot\gamma(t) = \max_{\|X\|\le\|\dot\gamma(t)\|} a(t)X$$

Subdifferentials – maximizers

For almost every $t \in \mathbb{R}$:

- $a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$
- $\tilde a$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$
- $\|\dot\gamma(t)\| = 1 = \|Y\|$

$$\tilde a Y = \max_{\|X\|\leq 1} \tilde a X = 1 \quad \text{and} \quad a(t)\dot\gamma(t) = \max_{\|X\|\leq 1} a(t)X = 1$$

Subdifferentials – maximizers

For almost every $t \in \mathbb{R}$:

- $a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$
- $\tilde a$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$
- $\|\dot\gamma(t)\| = 1 = \|Y\|$

$$\tilde a Y = \max_{\|X\|\le 1} \tilde a X = 1 \quad \text{and} \quad a(t)\dot\gamma(t) = \max_{\|X\|\le 1} a(t)X = 1$$

$$a(t)Y \equiv C = \tilde a Y \implies a(t)Y = \tilde a Y = 1.$$

Subdifferentials – maximizers

For almost every $t \in \mathbb{R}$:

- $a(t)$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $\dot\gamma(t)$
- $\tilde{a}$ is a subdifferential of $\frac{1}{2}\|\cdot\|^2$ at the point $Y = \lim_{k\to\infty} \dot\gamma(\lambda_k t)$
- $\|\dot\gamma(t)\| = 1 = \|Y\|$

$$\tilde{a}Y = \max_{\|X\|\le 1} \tilde{a}X = 1 \quad \text{and} \quad a(t)\dot\gamma(t) = \max_{\|X\|\le 1} a(t)X = 1$$

$$a(t)Y \equiv C = \tilde{a}Y \implies a(t)Y = \tilde{a}Y = 1.$$

$\|\cdot\|$ strictly convex $\implies$ $a(t)$ has a unique maximum
$\implies$ $Y = \dot\gamma(t)$ for almost every $t \in \mathbb{R}$
$\implies$ $\gamma\colon \mathbb{R} \to G$ is a line

Thank you for your attention!